ON THE SUPPORT OF DIFFUSION PROCESSES WITH APPLICATIONS TO THE STRONG MAXIMUM PRINCIPLE

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1. Introduction

Let $a\colon [0, \infty) \times R^d \to S_d$ and $b\colon [0, \infty) \times R^d \to R^d$ be bounded continuous functions, where S_d denotes the class of symmetric, nonnegative definite $d \times d$ matrices. From a and b form the operator

$$(1.1) \qquad L_t = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, x) \frac{\partial}{\partial x_i}.$$

A strong maximal principle for the operator $(\partial/\partial t) + L_t$ is a statement of the form: "for each open $\mathcal{G} \subseteqq [0, \infty) \times R^d$ and each $(t_0, x_0) \in \mathcal{G}$ there is a set $\mathcal{G}(t_0, x_0) \subseteqq \mathcal{G}$ with the property that $(\partial f/\partial t) + L_t f \geqq 0$ on $\mathcal{G}(t_0, x_0)$ and $f(t_0, x_0) = \sup_{\mathcal{G}(t_0,x_0)} f(t, x)$ imply $f \equiv f(t_0, x_0)$ on $\mathcal{G}(t_0, x_0)$." Of course, in order for a strong maximum principle to be very interesting it must describe the set $\mathcal{G}(t_0, x_0)$. Further, it should be possible to show that $\mathcal{G}(t_0, x_0)$ is maximal. That is, one wants to know that if $(t_1, x_1) \in \mathcal{G} - \mathcal{G}(t_0, x_0)$, then there is an f satisfying $(\partial f/\partial t) + L_t f \geqq 0$ on $\mathcal{G}$ (perhaps in a generalized sense) such that $f(t_0, x_0) = \sup f(t, x)$, and $f(t_1, x_1) < f(t_0, x_0)$.

In the case when $a(t, x)$ is positive definite for all (t, x), L. Nirenberg [6] has shown that $\mathcal{G}(t_0, x_0)$ can be taken as the closure in $\mathcal{G}$ of the set of $(t_1, x_1) \in \mathcal{G} \cap ([t_0, \infty) \times R^d)$ such that there exists a continuous map $\phi\colon [t_0, t_1] \to R^d$ with the properties that $\phi(t_0) = x_0$, $\phi(t_1) = x_1$, and $(t, \phi(t)) \in \mathcal{G}$ for all $t \in (t_0, t_1)$. We will give a probabilistic proof of the Nirenberg maximum principle in Section 3. Moreover, we will also prove there that Nirenberg's $\mathcal{G}(t_0, x_0)$ is maximal in the desired sense.

If a is only nonnegative definite, the problem of finding a suitable maximum principle is more difficult. Results in this direction have been proved by J.-M. Bony [1] and C. D. Hill [3]. Both of these authors employ a modification of the technique originally introduced by E. Hopf for elliptic operators and later adapted by Nirenberg for parabolic ones. The major drawback to Bony's

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