STOCHASTIC INTEGRALS AND PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

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1. Introduction

Let $X = \{X(t); 0 \leqq t \leqq 1\}$ be a real valued stochastic process with stationary independent increments and right continuous paths $X(0) = 0$. The characteristic function of $X(t)$ then has the form $\exp\{t\psi(u)\}$, where

$$(1.1) \qquad \psi(u) = iug - \tfrac{1}{2}\sigma^2 u^2 + \int \left[e^{iux} - 1 - \frac{iux}{1 + x^2} \right] \nu(dx).$$

The measure ν is called the Lévy measure of X, and ψ is called the exponent. It will be assumed throughout that $\sigma^2 = 0$. The index $\beta(X)$ of the process X is

$$(1.2) \qquad \beta(X) = \inf\left\{ p > 0; \int_{|x|<1} |x|^p \nu(dx) < \infty \right\}.$$

If $\int_{|x|<1} |x| \nu(dx) < \infty$, then by subtracting a linear term from X one may write the exponent ψ as

$$(1.3) \qquad \psi(u) = \int [1 - e^{iux}]\, \nu(dx);$$

it will be assumed from now on that the exponent is in this form whenever $\int_{|x|<1} |x| \nu(dx) < \infty$.

This paper studies the sample function behavior of processes $Y = \{Y(t); 0 \leqq t \leqq 1\}$, where $Y(t)$ has the form $Y(t) = \int_0^t v(s)\, dX(s)$ and where $v = \{v(s); 0 \leqq s \leqq 1\}$ is a stochastic process of a special type described below. Section 2 contains a development of the theory of such stochastic integrals, together with conventions and notations prerequisite for the rest of the paper. The construction of the stochastic integral is made to depend on an inequality of L. E. Dubins and J. L. Savage, and has applications to more general theories of stochastic integration. In Section 3 a local limit theorem is proved. If $|v(s)| \leqq 1$ and if $p > \beta(X)$, then $|Y(t)| t^{-1/p}$ converges to zero a.s. as $t \downarrow 0$. This generalizes (with different proof) a result known for the case $v(s) \equiv 1$ (see [2]). To state the results of Section 4, let $\pi_n: 0 = t_{n,1} < \cdots < t_{n,k_n} = 1$ be a sequence of partitions of $[0, 1]$ satisfying

$$(1.4) \qquad \lim_{n \to \infty} \max_k [t_{n,k+1} - t_{n,k}] = 0;$$

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