# REGENERATIVE PHENOMENA AND THE CHARACTERIZATION OF MARKOV TRANSITION PROBABILITIES

J. F. C. KINGMAN
OXFORD UNIVERSITY

## 1. Markov chains in continuous time

It is the object of this paper to draw together certain lines of research which during the last decade have grown out of the problem of characterizing the functions which can arise as transition probabilities of continuous time Markov chains. This problem is now solved (see Sections 8 and 9), although as usual its solution has thrown up further problems which demand attention.

The evolution of a Markov chain $X_t$ in continuous time, with stationary transition probabilities, on a countable state space $S$, is as usual [2] described by the functions

$$p_{i,j}(t) = \mathbf{P}(X_{s+t} = j \mid X_s = i) \tag{1.1}$$

for $i, j \in S$, $t > 0$. These necessarily satisfy the conditions

$$p_{i,j}(t) \geqq 0, \qquad \sum_{j \in S} p_{i,j}(t) = 1, \tag{1.2}$$

and

$$p_{i,j}(t + u) = \sum_{k \in S} p_{i,k}(t) p_{k,j}(u), \tag{1.3}$$

to which it is usual to add the continuity condition

$$\lim_{t \to 0} p_{i,j}(t) = p_{i,j}(0) = \delta_{i,j}. \tag{1.4}$$

Conversely, given any array $(p_{i,j}; i, j \in S)$ of functions satisfying (1.2) and (1.3), a Markov chain $X_t$ can be constructed so as to satisfy (1.1).

It is therefore not surprising that a substantial part of the theory of Markov chains should be concerned with the consequences of (1.2), (1.3), and (1.4) for the functions $p_{i,j}$. It is possible to regard this as a problem in pure analysis, but those methods that have proved most powerful have had strong probabilistic motivation. The following list of typical results, taken from [2], will illustrate the achievements of this part of the theory (they are arranged in roughly increasing order of difficulty):