APPROXIMATION OF CONTINUOUS ADDITIVE FUNCTIONALS

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1. Introduction

The purpose of this exposition is to give correct proofs of two well known and reasonably important propositions concerning continuous additive functionals. We adopt the terminology and notation of [1] throughout. We fix once and for all a standard process $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ with state space E. (See (I–9.2); all such references are to [1].)

The following two theorems are important facts about continuous additive functionals (CAF's) of such a process. (See (IV–2.21) or [2].)

THEOREM 1. *Let A be a* CAF *of X. Then $A = \Sigma_{n=1}^{\infty} A^n$ where each A^n is a* CAF *of X having a bounded one potential.*

Making use of Theorem 1, one can establish the following result. (See (V–2.1) or [2].)

THEOREM 2. *Suppose that X has a reference measure (that is, satisfies the hypothesis of absolute continuity). Then every* CAF *of X is equivalent to a perfect* CAF.

Unfortunately, the proofs known to me of Theorem 1 are not convincing. For example, the "proof" in [1] goes as follows. Let A be a CAF of X. Define

$$\varphi(x) = E^x \int_0^\infty e^{-t} e^{-A_t}\, dt. \tag{1.1}$$

Clearly, $0 < \varphi \leqq 1$ and φ is universally measurable; actually it is not difficult to see that φ is nearly Borel, but this is not required. Let $R = \inf\{t : A_t = \infty\}$. Then it is easy to check that R is a terminal time and that $P^x(R > 0) = 1$ for all x. Obviously, $\varphi(x) = E^x \int_0^R e^{-t} e^{-A_t}\, dt$. Now if T is any stopping time,

$$\begin{aligned} E^x\{e^{-T}\varphi(X_T); T < R\} &= E^x\left\{e^{-T} \int_0^{R\circ\theta_T} e^{-t} e^{-A_t\circ\theta_T}\, dt; T < R\right\} \\ &= E^x\left\{e^{A_T} \int_T^R e^{-u} e^{-A_u}\, du; R < T\right\}, \end{aligned} \tag{1.2}$$

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