LOGARITHMIC POTENTIALS AND PLANAR BROWNIAN MOTION

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In this paper we continue our discussion of the connection between potential theory and Brownian motion begun in "Classical Potential Theory and Brownian Motion" that also appears in this Symposium volume. Throughout this paper, we will be dealing with a two dimensional Brownian motion process. We will continue numbering the sections from where we left off in the previous paper.

8. Planar Brownian motion

In Section 5, we saw that for a Brownian motion process in $n \geqq 3$ dimensions, $P_x(\lim_{t\to\infty} |X_t| = \infty) = 1$ for all x. In sharp contrast to this situation, a planar Brownian motion is certain to hit any nonpolar set.

THEOREM 8.1. *Let B be a Borel set. Then $P_x(V_B < \infty)$ is either identically* 1 *or identically* 0.

PROOF. A simple computation shows that for any $x \in R^2$, $\int_0^t p(s, x)\, ds \uparrow \infty$ as $t \uparrow \infty$. Thus, for any nonnegative function f having nonzero integral,

$$\lim_{t\to\infty} \int_0^t P^s f(x)\, ds = \infty. \tag{8.1}$$

Let $\varphi(x) = P_x(V_B < \infty)$. Then for any $h > 0$,

$$0 \leqq \int_0^t P^s(\varphi - P^h\varphi)\, ds = \int_0^h P^s\varphi\, ds - \int_t^{t+h} P^s\varphi\, ds \leqq 2h. \tag{8.2}$$

Letting $t \uparrow \infty$, we see that

$$0 \leqq \int_0^\infty P^s(\varphi - P^h\varphi)\, ds \leqq 2h. \tag{8.3}$$

But then it must be that $\varphi = P^h\varphi$ a.e. Since $P^t\varphi \uparrow \varphi$ as $t \downarrow 0$ and $P^t(P^h\varphi) \uparrow P^h\varphi$ as $t \downarrow 0$, it follows that $\varphi(x) = P^h\varphi(x)$ for all x. Using Proposition 2.3, we see that $\varphi(x) \equiv \alpha$ for some constant α. Now

$$P_x(t < V_B < \infty) = \int_{R^2} q_B(t, x, y)\varphi(y)\, dy = \alpha P_x(V_B > t). \tag{8.4}$$

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