CLASSICAL POTENTIAL THEORY AND BROWNIAN MOTION

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1. Introduction

Pioneering work of Doob, Kac, and Kakutani showed that there was a beautiful and deep connection between certain problems in the study of Brownian motion and those of classical potential theory. This work stimulated much research on the theory of Markov processes. In spite of all this work, however, there doesn't appear anywhere in the literature any reasonably complete treatment of the connection between potential theory and Brownian motion. In this paper and its companion "Logarithmic Potentials and Planar Brownian Motion" which follows in this volume, we present this connection in a way that is both elementary and essentially selfcontained. Our treatment here is not complete and will be expanded upon in a forthcoming monograph.

This paper, being basically expository in nature, contains essentially nothing new. Its novelty (if any) consists in the treatment given to the topics discussed. In one place, however, we do seem to have a result that is new. This is in finding all bounded solutions of the modified Dirichlet problem for any arbitrary open set G, and in giving a necessary and sufficient condition for there to be a unique such solution.

In this paper, we consider a Brownian motion process X_t in $n \geqq 2$ dimensional Euclidean space R^n. Let B be a Borel set and set

$$V_B = \inf\{t > 0: X_t \in B\}, \qquad V_B = \infty \text{ if } X_t \notin B \text{ for all } t > 0. \tag{1.1}$$

In Section 2, we present preliminary facts about Brownian motion that are needed to develop classical potential theory from a probabilistic point of view. A set B is called polar if $P_x(V_B < \infty) \equiv 0$. A point x is called regular for B if $P_x(V_B = 0) = 1$. In Section 3, we prove that the set of points in B that are not regular for B is a polar set. In this section, we also show that points are polar and gather together a few more facts of a technical nature that are needed for work in the later sections. The Dirichlet problem for an arbitrary open set G is discussed in Section 4.

Starting with Section 5 and throughout the remainder of this paper, we assume that we are dealing with Brownian motion in $n \geqq 3$ dimensions. (The planar

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