THE STRUCTURE OF A MARKOV CHAIN

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1. Introduction

Let p be a standard transition function on the set I of integers, that is, a function from $(0, \infty) \times I \times I$ into $[0, 1]$ satisfying

$$\sum_j p(t, i, j) = 1, \qquad p(s + t, i, k) = \sum_j p(s, i, j)p(t, j, k), \tag{1.1}$$

together with the continuity condition $\lim_{t \to 0} p(t, i, i) = 1$. Let f be an absolute probability function, that is, a function from $(0, \infty) \times I$ into $[0, 1]$, satisfying

$$\sum_j f(t, j) = 1, \qquad f(s + t, j) = \sum_i f(s, i)p(t, i, j). \tag{1.2}$$

Let L be an arbitrary set containing I as a subset. There is then a Markov process $\{x(t), t > 0\}$ with state space L having the specified transition and absolute probability functions. The notation $x(t)$ will always refer to the tth random variable of such a process, and the process will be called "smooth" if L is topological and if almost every sample function is right continuous with left limits on $(0, \infty)$. Note that this condition does not require the existence of a right limit at 0. For each $t > 0$ the random variable $x(t)$ almost surely has its values in I, but it has been known since Ray's work [11] in 1959 that L and the process can be chosen to make the process and properly chosen extensions of the transition function have desirable smoothness properties. One can always choose L to be an entrance space in the sense of [4]; that is, one can choose L to satisfy the following conditions:

(a) L is a Borel subset of a compact metric space in which I is dense;

(b) for every absolute probability function f there is a smooth corresponding process with state space L;

(c) for every integer j, $p(\cdot, \cdot, j)$ has a continuous extension to $(0, \infty) \times L$ and (1.1) is satisfied for i allowed to be any point of L;

(d) if ξ is in L and if $\{x(t), t > 0\}$ is a smooth process with absolute probability function given by $f(t, i) = p(t, \xi, i)$, then $x(0+)$ exists (and is in L) almost surely.

In the following, i, j, k are integers and ξ, η are points of a specified entrance space. The probability measure determined by a smooth process with $f(t, i) =$