ON THE LAW OF THE ITERATED LOGARITHM FOR MAXIMA AND MINIMA

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1. Introduction and summary

Let $w(t)$, $0 \leqq t \leqq \infty$, denote a standard Wiener process. The general law of the iterated logarithm (see [6], p. 21) says that if g is a positive function such that $g(t)/\sqrt{t}$ is ultimately nondecreasing, then

$$P\{w(t) \geqq g(t) \text{ i.o. } t \uparrow \infty\} \tag{1.1}$$

equals 0 or 1, according as

$$\int_1^\infty \frac{g(t)}{t^{3/2}} \exp\left\{-\frac{1}{2}\frac{g^2(t)}{t}\right\} dt < \infty \text{ or } = \infty. \tag{1.2}$$

(The notation i.o. $t \uparrow \infty$ ($t \downarrow 0$) means for arbitrarily large (small) t.) In particular, for $k \geqq 3$ and

$$g(t) = \left[2t\left(\log_2 t + \tfrac{3}{2}\log_3 t + \sum_{i=4}^{k} \log_i t + (1+\delta)\log_{k+1} t\right)\right]^{1/2}, \tag{1.3}$$

the probability (1.1) is 0 or 1 according as $\delta > 0$ or $\delta \leqq 0$. (We write $\log_2 = \log\log$, $e_2 = e^e$, and so on.)

For applications in statistics it is of interest to compute as accurately as possible

$$P\{w(t) \geqq g(t) \text{ for some } t \geqq \tau\} \tag{1.4}$$

for functions g for which this probability is < 1; that is, functions for which (1.2) converges (see [3], [10], [12]). In [11], we gave a method for computing (1.4) exactly for a certain class of functions g. A sketch of this method follows. Since $\exp\{\theta w(t) - \frac{1}{2}\theta^2 t\}$, $0 \leqq t < \infty$, is a martingale for each θ, Fubini's theorem shows that $\int_0^\infty \exp\{\theta w(t) - \frac{1}{2}\theta^2 t\}\, dF(\theta)$, $0 \leqq t < \infty$, is also a martingale for any σ-finite measure F on $(0, \infty)$. Let

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