THE RANGE OF RANDOM WALK

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1. Introduction

Let $\{X_n, n \geqq 1\}$ be a sequence of independent identically distributed random variables, defined on a probability space $(\Omega, \mathscr{F}, P)$, which take values in the d dimensional integer lattice E_d. The sequence $\{S_n, n \geqq 0\}$ defined by $S_0 = 0$ and $S_n = \Sigma_{k=1}^n X_k$ is called a random walk. The range of the random walk, denoted by R_n, is the cardinality of the set $\{S_0, S_1, \cdots, S_n\}$; it is the number of distinct points visited by the random walk up to time n. Our object here is to study the asymptotic behavior of R_n. Two specific problems are considered:

(i) Does $R_n/ER_n \to 1$ a.s.? If so, this will be called the strong law for R_n.

(ii) Does $(R_n - ER_n)(\text{Var } R_n)^{-1/2}$ converge in distribution? If so, this will be called the central limit theorem for R_n.

The random walk may take place on a proper subgroup of E_d. In this case, the subgroup is isomorphic to some E_k for $k \leqq d$; if $k < d$, then the transformation should be made and the problem considered in k dimensions. We will assume throughout the paper that this reduction has been made, if necessary, and that d is the genuine dimension of the random walk.

Dvoretzky and Erdös [2] proved the strong law for the range of simple random walk for $d \geqq 2$. (Simple random walk is one for which the distribution of X_1 assigns probability $(2d)^{-1}$ to each of the $2d$ neighbors of the origin.) Their method was to obtain a somewhat crude estimate of Var R_n and then use the Chebyshev inequality. While this worked fairly easily for $d \geqq 3$, they had to work much harder for $d = 2$. By a rather sophisticated technique, they managed to improve the required probability estimate enough to obtain the proof.

Let $p = P[S_1 \neq 0, S_2 \neq 0, \cdots]$. The random walk is called transient if $p > 0$ and recurrent otherwise. Using a very elegant technique Kesten, Spitzer, and Whitman ([12], p. 38) proved that for *all* random walks $R_n/n \to p$ a.s. For transient random walks $ER_n \sim pn$, so that their result includes the strong law for all transient random walks.

There are recurrent random walks only if the dimension is one or two. In [7], we attempted to prove the strong law for R_n for the general recurrent random walk in two dimensions, but we succeeded only partially. Our method there was to imitate the proof of Dvoretzky and Erdös, that is, to obtain an estimate for Var R_n and then to improve the probability estimate by their methods. In Section 3, we will prove the strong law for R_n for *all* two dimensional recurrent random walks by an essentially different technique. We use a very delicate

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