A BOUND FOR THE ERROR IN THE NORMAL APPROXIMATION TO THE DISTRIBUTION OF A SUM OF DEPENDENT RANDOM VARIABLES

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1. Introduction

This paper has two aims, one fairly concrete and the other more abstract. In Section 3, bounds are obtained under certain conditions for the departure of the distribution of the sum of n terms of a stationary random sequence from a normal distribution. These bounds are derived from a more abstract normal approximation theorem proved in Section 2. I regret that, in order to complete this paper in time for publication, I have been forced to submit it with many defects remaining. In particular the proof of the concrete results of Section 3 is somewhat incomplete.

A well known theorem of A. Berry [1] and C-G. Esséen [2] asserts that if $X_1, X_2, \cdots$ is a sequence of independent identically distributed random variables with $EX_i = 0, EX_i^2 = 1$, and $\beta = E|X_i|^3 < \infty$, then the cumulative distribution function of $(1/\sqrt{n})\ \Sigma_{i=1}^{n} X_i$ differs from the unit normal distribution by at most $K\beta/\sqrt{n}$ where K is a constant, which can be taken to be 2. It seems likely, but has never been proved and will not be proved here, that a similar result holds for stationary sequences in which the dependence falls off sufficiently rapidly and the variance of $(1/\sqrt{n})\ \Sigma_{i=1}^{n} X_i$ approaches a positive constant. I. Ibragimov and Yu. Linnik ([3], pp. 423–432) prove that, under these conditions, the limiting distribution of $(1/\sqrt{n})\ \Sigma X_i$ is normal with mean 0 and a certain variance σ^2. Perhaps the best published results on bounds for the error are those of Phillip [5], who shows that if in addition the X_i are bounded, with exponentially decreasing dependence, then the discrepancy is roughly of the order of $n^{-1/4}$. In Corollary 3.2 of the present paper it is proved that under these conditions the discrepancy is of the order of $n^{-1/2}(\log n)^2$. Actually the assumption of boundedness is weakened to the finiteness of eighth moments. In Corollary 3.1 it is proved that if the assumption of exponential decrease of dependence is strengthened to m dependence, the error in the normal approximation is of the order of $n^{-1/2}$.

The abstract normal approximation theorem of Section 2 is elementary in the sense that it uses only the basic properties of conditional expectation and the elements of analysis, including the solution of a first order linear differential equation. It is also direct, in the sense that the expectation of a fairly arbitrary