RECENT RESULTS ON REFINEMENTS OF THE CENTRAL LIMIT THEOREM

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1. Introduction and summary

Let $\{X_n\}$ be a sequence of independent and identically distributed (i.i.d.) random vectors in R^k with zero mean vector and identity covariance matrix. The distribution Q_n of the normalized sum $n^{-1/2}(X_1 + \cdots + X_n)$ converges weakly to the k dimensional standard normal distribution Φ. Although many important results on rates of convergence have been obtained in the past, most of them refer to approximations of the distribution function F_n of Q_n by the normal distribution function. An exception to this is the case where Q_n is assumed to have a density with respect to Lebesgue measure or to have a lattice distribution. In this situation, one obtained local limit theorems as well (see [14], Chapter 16, and [19]). The first notable exception was a result of Esseen [13] which states that *if fourth moments are finite, then, uniformly over all spheres* S (*open or closed*) *with center at the origin, one has*

$$Q_n(S) - \Phi(S) = O(n^{-k/(k+1)}), \qquad n \to \infty. \tag{1.1}$$

Esseen showed the remarkable depth of this result by relating a special case of this to the *lattice point problem* of analytic number theory. In 1960, Ranga Rao [29] investigated the rate of convergence over the class $\mathscr{C}$ of all measurable convex sets and proved that *if fourth moments are finite, then*

$$\sup_{C \in \mathscr{C}} |Q_n(C) - \Phi(C)| = O\left(n^{-1/2}(\log n)^{(k-1)/2(k+1)}\right), \qquad n \to \infty. \tag{1.2}$$

He also obtained a number of asymptotic expansions extending some results of Cramér [9] (Chapter 7) and Esseen [13] for distribution functions. The present author [1] and von Bahr [34] independently obtained rates of convergence for general classes of sets; a typical application gives the following precise bound for $\mathscr{C}$:

$$\sup_{C \in \mathscr{C}} |Q_n(C) - \Phi(C)| = O(n^{-1/2}), \qquad n \to \infty. \tag{1.3}$$

In [1], this was proved under the assumption of finiteness of moments of order $3 + \delta$ for some positive δ, while in [34] $E|X_1|^{k+1}$ was assumed finite for $k \geqq 2$.

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