SAMPLE BEHAVIOR OF GAUSSIAN PROCESSES

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1. Introduction

Doob remarked in his 1953 book that, "very few facts specifically true of Gaussian processes are known." This statement is no longer true; the field is very active, and Gaussian processes now form a very special class. The various zero-one laws discovered for Gaussian processes show that their sample functions behave almost deterministically. Indeed, the deterministic-like properties of Gaussian models even appear nonphysical. (A well-known example is the property of the Wiener and other Gaussian processes that the sample quadratic variation on an interval is constant.) However, we will not pursue this point.

This work surveys some recent results on sample behavior of Gaussian processes and continues the study of the supremum $\|X\|$ of a bounded Gaussian process $X(t)$. It is proved in particular that

$$\lim_{t\to\infty} \frac{1}{t^2} \log P(\|X\| > t) = -(2\sigma^2)^{-1}, \tag{1.1}$$

where σ^2 is the supremum of the variances of the individual $X(t)$. This extends work of Fernique [10] and Landau and Shepp [17].

2. A survey of sample behavior

The question due to Kolmogorov (see [7], which stimulated much of the interest in this area, asks which stationary Gaussian processes have continuous sample paths. A stationary process X has covariance $R(s, t) = R(s - t) = EX(s)X(t)$. The question is then for which nonnegative definite functions R is X a.s. continuous. It is, of course, necessary that R be continuous, and at first it is surprising that this is not also sufficient. Kolmogorov must have had examples of stationary processes with discontinuous paths, probably based on random Fourier series.

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