STATISTICS OF CONDITIONALLY GAUSSIAN RANDOM SEQUENCES

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1. Statement of the problem

We are given, on a probability space $(\Omega, \mathscr{F}, P)$, a random sequence (θ_t, ξ_t),

$$\theta_t = (\theta_1(t), \cdots, \theta_k(t)), \xi_t = (\xi_1(t), \cdots, \xi_\ell(t)), \tag{1.1}$$

for $t = 0, 1, \cdots$, defined by the system of recursive equations

$$\begin{aligned} \theta_{t+1} &= a_0(t, \omega) + a_1(t, \omega)\theta_t + b(t, \omega)\Delta_1(t + 1), \\ \xi_{t+1} &= A_0(t, \omega) + A_1(t, \omega)\theta_t + B(t, \omega)\Delta_2(t + 1), \end{aligned} \tag{1.2}$$

where $\Delta_1(t)$ and $\Delta_2(t)$ are Gaussian and, in general, mutually dependent random vectors; while the vectors $a_i(t, \omega)$ and $A_i(t, \omega)$ and the matrices $b(t, \omega)$ and $B(t, \omega)$ are, for each t, $\mathscr{F}_t^\xi = \sigma\{\omega: \xi_0, \cdots, \xi_t\}$ measurable. The system (1.2) is to be solved for the initial conditions (θ_0, ξ_0), which are assumed to be independent of the processes $\Delta_1(t)$ and $\Delta_2(t)$, $t = 0, 1, \cdots$.

In the sequel, θ_t will be treated as a vector with unobservable components, and ξ_t as a vector with observable components. The statistical problems we wish to consider involve the construction of optimal (in the mean square sense) estimates of the unobservable process θ_t in terms of observations on the process ξ_t.

One can distinguish the following three basic problems of estimation, which will be called the problems of filtering, interpolation, and extrapolation.

Filtering. By filtering is understood the problem of estimating the unobservable vector θ_t by means of observations on the values of $\xi^t = (\xi_0, \xi_1, \cdots, \xi_t)$. We put $\Pi_\alpha(t) = P\{\theta_t \leqq \alpha | \mathscr{F}_t^\xi\}$ (where for vectors $x = (x_1, \cdots, x_k)$, $y = (y_1, \cdots, y_k)$ the inequality $x \leqq y$ is taken to mean that $x_i \leqq y_i$ for all $i = 1, \cdots, k$), let $m(t) = \mathbf{M}(\theta_t | \mathscr{F}_t^\xi)$, and

$$\gamma(t) = \operatorname{Cov}(\theta_t | \mathscr{F}_t^\xi) = \mathbf{M}\{(\theta_t - m(t))(\theta_t - m(t))^* | \mathscr{F}_t^\xi\}. \tag{1.3}$$

It is well known that $m(t) = \mathbf{M}(\theta_t | \mathscr{F}_t^\xi)$ is the optimal (in the mean square sense) estimate of θ_t by means of $\xi^t = (\xi_0, \xi_1, \cdots, \xi_t)$, and that

$$\operatorname{tr} \mathbf{M}\gamma(t) = \sum_{i=1}^{k} \mathbf{M}[\theta_i(t) - m_i(t)]^2, \tag{1.4}$$

where $m(t) = (m_1(t), \cdots, m_k(t))$ is the error corresponding to this estimate.