ON THE ROOT PROBLEM IN ERGODIC THEORY

DONALD S. ORNSTEIN
STANFORD UNIVERSITY

1. Introduction

One of the unsolved problems in ergodic theory is the following. Let T be an invertible measure preserving transformation on the unit interval. When does T have a square root? When can T be imbedded in a flow? In his book on ergodic theory, Halmos asked, (1) if every weakly mixing transformation had a square root, (2) if every Bernoulli shift had a square root, and (3) if every Bernoulli shift could be imbedded in a flow. Chacon [1] showed that the answer to (1) was negative. We showed [5], [6] that the answer to (2) and (3) was yes. These results seem to indicate that "enough mixing" forces T to have a square root or to be imbeddable in a flow.

It is the purpose of this paper to give an example of a mixing transformation that has no square root. (T is mixing if and only if

$$\lim_{n\to\infty} m(T^n A \cap B) = m(A)m(B) \tag{1.1}$$

where $m(A)$ denotes the measure of the set A.) The transformation T that we will construct will not only lack a square root but will have the property that if S is a measure preserving transformation of the unit interval such that $ST = TS$, then $S = T^i$ for some integer i (possibly negative or 0).

It is still not known if every K automorphism has a square root. (K automorphisms have a stronger mixing property than "mixing." It was once conjectured that all K automorphisms were Bernoulli shifts but this is now known to be false [7].)

Before starting the construction of our example we shall prove the following theorem which we believe is of independent interest.

THEOREM 1.1. *If T is a measure preserving invertible transformation of* (0, 1) *such that every power of T is ergodic, and if T has the property that there is a constant K, $K > 1$, and* $\limsup_{n\to\infty} m(T^n A \cap B) < Km(A)m(B)$ *for all measurable sets A and B, then T is mixing.*

We could construct our example without the help of the above theorem but only at the cost of considerable additional complication.

The main motivation for proving Theorem 1.1, however, comes from the following conjecture of Kakutani. If there is a constant K, with $0 < K < 1$, such

This research was supported in part by a grant from the National Science Foundation under GP-8781.