ON UNIQUE ERGODICITY

WOLFGANG KRIEGER
OHIO STATE UNIVERSITY

1. Introduction

A homeomorphism U of a compact metric space onto itself is said to be uniquely ergodic if it possesses a unique invariant Borel probability measure μ_U. For an introduction to the theory of unique ergodicity, we refer the reader to J. Oxtoby [11]. A point x in the shift space $\Omega^{\mathbf{Z}}$, where Ω is a finite state space, is called a uniquely ergodic sequence if the shift S, $(Sx)_i = x_{i+1}$, where $i \in \mathbf{Z}$, $x \in \Omega^{\mathbf{Z}}$, is a uniquely ergodic homeomorphism of the orbit closure $\mathcal{O}_x = \overline{\{S^i x : i \in \mathbf{Z}\}}$ of x. We denote the shift invariant probability measure of a uniquely ergodic sequence x by μ_x.

S. Kakutani [7], M. Keane [8], and K. Jacobs and M. Keane [5] have constructed a variety of uniquely ergodic sequences and investigated their measure theoretic properties. The first examples of weakly mixing uniquely ergodic systems were given by Jacobs [3]. F. Hahn and Y. Katznelson [2] constructed uniquely ergodic sequences with arbitrarily high entropy and Ch. Grillenberger [1] produced uniquely ergodic sequences in $\Omega^{\mathbf{Z}}$ whose entropy is arbitrarily close to $\log |\Omega|$. Further constructions of uniquely ergodic sequences were given by W. Veech (Section 3 of [12]).

We shall prove in Section 3 that for every ergodic shift invariant measure μ on $\Omega^{\mathbf{Z}}$ whose entropy $h(\mu)$ is less than $\log |\Omega|$, there exists a uniquely ergodic sequence $x \in \Omega^{\mathbf{Z}}$ such that the systems $(\Omega^{\mathbf{Z}}, \mu, S)$ and $(\mathcal{O}_x, \mu_x, S)$ are isomorphic and such that μ_x is in any given weak neighborhood of μ.

This result and the finite generator theorem for ergodic measure preserving transformations (see [9] and [10]) imply that every ergodic measure preserving invertible transformation T of a Lebesgue measure space with finite entropy $h(T)$ is isomorphic to a system $(\mathcal{O}_x, \mu_x, S)$, where x is a uniquely ergodic sequence in $\Omega^{\mathbf{Z}}$ and $\exp \{h(T)\} < |\Omega| \leqq \exp \{h(T)\} + 1$. In Section 4, we shall show that every ergodic invertible measure preserving transformation T of a Lebesgue measure space is isomorphic to a system (U, C, μ_U), where U is a uniquely ergodic homeomorphism of the Cantor discontinuum C. This was recently established by R. Jewett [6] under the additional assumption that T be weakly mixing, and conjectured by him to hold in the ergodic case. Our method of proof combines the basic idea of Jewett with the methods that were developed for the proof of the finite generator theorem for ergodic measure preserving transformations (see [9] and [10]). We require some tools that we develop in Section 2.

Jacobs has recently shown that every weakly mixing flow on a Lebesgue measure space is isomorphic to a flow of homeomorphisms of a compact metric space together with a unique invariant Borel measure [4].