# ON THE FOUNDATIONS OF COMBINATORIAL THEORY (VI): THE IDEA OF GENERATING FUNCTION

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## 1. Introduction

Since Laplace discovered the remarkable correspondence between set theoretic operations and operations on formal power series, and put it to use with great success to solve a variety of combinatorial problems, generating functions (and their continuous analogues, namely, characteristic functions) have become an essential probabilistic and combinatorial technique. A unified exposition of their theory, however, is lacking in the literature. This is not surprising, in view ot the fact that all too often generating functions have been considered to be simply an application of the current methods of harmonic analysis. From several of the examples discussed in this paper it will appear that this is not the case: in order to extend the theory beyond its present reaches and develop new kinds of algebras of generating functions better suited to combinatorial and probabilistic problems, it seems necessary to abandon the notion of group algebra (or semigroup algebra), so current nowadays, and rely instead on an altogether different approach.

The insufficiency of the notion of semigroup algebra is clearly seen in the example of Dirichlet series. The functions

$$n \to 1/n^s \tag{1.1}$$

defined on the semigroup $S$ of positive integers under multiplication, are characters of $S$. They are not, however, all the characters of this semigroup, nor does there seem to be a canonical way of separating these characters from the rest (see, for example, Hewitt and Zuckerman [32]). In other words, there does not seem to be a natural way of characterizing the algebra of formal Dirichlet series as a subalgebra of the semigroup algebra (eventually completed under a suitable topology) of the semigroup $S$. In the present theory, however, the algebra of formal Dirichlet series arises naturally from the incidence algebra (definition below) of the lattice of finite cyclic groups, as we shall see.