INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS OF OPERATORS ON MARTINGALES

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1. Introduction

Let $\mathscr{M}$ be a family of martingales on a probability space $(\Omega, \mathscr{A}, P)$ and let Φ be a nonnegative function on $[0, \infty]$. The general question underlying both [2] and the present work may be stated as follows: *If U and V are operators on $\mathscr{M}$ with values in the set of nonnegative $\mathscr{A}$ measurable functions on Ω, under what further conditions does*

$$\lambda^{p_0} P(Vf > \lambda) \leqq c\|Uf\|_{p_0}^{p_0}, \qquad \lambda > 0, f \in \mathscr{M}, \tag{1.1}$$

imply $E\Phi(Vf) \leqq cE\Phi(Uf), f \in \mathscr{M}$? Here E denotes expectation, integration over Ω with respect to P, and the letter c denotes a positive real number, not necessarily the same number from line to line. In most applications, the first inequality can be proved easily for only one particular value of p_0, usually for $p_0 = 2$, although it is the second inequality that is really needed. Therefore, it is important to know conditions under which the second follows from the first.

In [2], the function Φ may be any nondecreasing function that satisfies a mild growth condition. The above question is then answered by suitably restricting the martingale f. In this paper, Φ is restricted to be convex, but no conditions are placed on the martingale f.

We state our main results in Section 2. Here, we mention one special but important application. If $f = (f_1, f_2, \cdots)$ is a martingale, we write

$$\begin{aligned} f_n &= \sum_{k=1}^{n} d_k, \qquad n \geqq 1, \\ f^* &= \sup_n |f_n|, \\ S(f) &= \left(\sum_{k=1}^{\infty} d_k^2\right)^{1/2} \end{aligned} \tag{1.2}$$

The maximal function f^* and the square function $S(f)$ are closely linked.

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