ON POISSON LAWS AND RELATED QUESTIONS

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1. Definitions and some lemmas

We shall consider a class of infinitely divisible laws, which may be called Poisson laws, defined on the Borel sets of a locally compact group. This class of probability measures is arrived at quite naturally by looking at the classical Poisson laws over the Borel sets $\mathscr{B}_1$ of one dimensional Euclidean space R_1. Let $a > 0$. Then the standard Poisson law with mean a can be written in the form

$$\exp\{a(\delta_1 - \delta_0)\} = \delta_0 + \sum_{k=1}^{\infty} \frac{a^k}{k!}(\delta_1 - \delta_0)^k, \tag{1.1}$$

where δ_x is the Dirac measure at $x \in R_1$. Multiplication of measures means convolution. Convergence of the series means convergence in norm. The measure $a(\delta_1 - \delta_0)$ obviously satisfies the following conditions. If f belongs to the set $C(R_1)$ of bounded, continuous functions and fulfills the conditions $f \geqq 0$ and $f(0) = 0$, then $a(\delta_1 - \delta_0)(f) \geqq 0$. Moreover, $a(\delta_1 - \delta_0)(1) = 0$, where 1 denotes the function f identically equal to 1. It is well known that more general probability laws of the Poisson type may be defined along these lines. Let v be any bounded Radon measure defined over $\mathscr{B}_1$ and satisfying the conditions $v(1) = 0$ and $v(f) \geqq 0$ for every $f \in C(R_1)$ with $f \geqq 0$ and $f(0) = 0$. Then e^v is a probability law of Poisson type. Note that 0 is the neutral element of the additive group of R_1, and that the set $\{0\}$ is a compact subgroup of R_1. These considerations lead easily to a generalization of Poisson laws on arbitrary, locally compact groups. To achieve this, some simple definitions are needed.

DEFINITION 1.1. *The set of all bounded Radon measures defined over the Borel sets $\mathscr{B}$ of a locally compact group G is denoted by $\mathscr{R}(G)$, or just by $\mathscr{R}$. The subset of all probability measures is denoted by $Z(G)$, or just by Z. If m is any measure in $\mathscr{R}$, then $S(m)$ denotes its support.*

DEFINITION 1.2. *A measure $\mu \in Z$ is said to be infinitely divisible if for every natural number n there exists a $\mu_{1/n} \in Z$ which satisfies the equation $\mu_{1/n}^n = \mu$. The measure $\mu_{1/n}$ is called an nth root of μ.*

DEFINITION 1.3. *Let H be an arbitrary compact subgroup of G. Then e_H denotes that probability measure belonging to $Z(G)$ whose restriction to $H \cap \mathscr{B}$ is the Haar measure; $\mathscr{R}_H$ denotes the set of all $m \in \mathscr{R}$ which satisfy the equation $e_H m = m e_H = m$. The set $Z \cap \mathscr{R}_H$ is denoted by Z_H.*