A COUNTEREXAMPLE ON MEASURABLE PROCESSES

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In 1947, Doob [7] posed the following question. Suppose $x = \{x_t, 0 \leqq t \leqq 1\}$ is a (jointly) measurable stochastic process with values in a compact space K, for example, the one point compactification $\bar{R}$ of the real line R. Let $\bar{P}_x$ be the distribution of x in the compact function space of all functions from $[0, 1]$ into K, where $\bar{P}_x$ is a regular Borel measure [15]. Then is the evaluation map $E: (t, f) \to f(t)$ necessarily measurable for the product measure $\lambda \times \bar{P}_x$, where λ is Lebesgue measure? I shall give a counterexample, assuming the continuum hypothesis. The counterexample is a Gaussian process. Replacing $([0, 1], \lambda)$ by an equivalent measure space (H, μ), where H is a Hilbert space and μ a suitable Gaussian probability measure, we can take the process x to be the standard Gaussian linear process L on H. Although we shall carry through the details only for this particular process, the method is applicable to various other processes represented by convergent series $\Sigma y_n(t)z_n(\omega)$ with independent terms such that $\Sigma y_n(t)$ and $\Sigma z_n(\omega)$ are not convergent in general. The possibility of weakening the continuum hypothesis assumption will be discussed in an Appendix.

Earlier, M. Mahowald [14] proposed a positive solution to the Kakutani-Doob problem. But the last step in his argument applies the Fubini theorem to sets in a product space which have not been shown to be measurable.

After the counterexample (Proposition 1), we give a few easier facts which also contribute to a broader conclusion that uncountable Cartesian products of compact metric spaces (for example, intervals) are relatively "bad" spaces as regards measurability.

DEFINITION. *Let $(X, \mathscr{B})$ be a measurable space. An X valued stochastic process with parameter set T and probability space $(\Omega, \mathscr{S}, P)$ is a function x from $T \times \Omega$ into X such that for each t in T, $x(t, \cdot)$ is measurable from $(\Omega, \mathscr{S})$ into $(X, \mathscr{B})$.*

Let X^T denote the set of all functions from T into X. Suppose X is a Polish space (complete separable metric space) or a compact Hausdorff space and $\mathscr{B}$ its class of Borel sets. Then for any stochastic process x as in Definition 1, there is a probability measure P_x on X^T such that for any $t_1, \cdots, t_n \in T$ and $B_1, \cdots, B_n \in \mathscr{B}$,

$$(1) \qquad P\{\omega: X(t_j, \omega) \in B_j, j = 1, \cdots, n\} = P_x\{f: f(t_j) \in B_j, j = 1, \cdots, n\},$$

according to a well-known theorem of Kolmogorov.

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