ON CONTINUOUS COLLECTIONS OF MEASURES

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1. Introduction

In this paper we will consider a problem whose formulation in probability terms is essentially as follows: when can one construct a stochastic process $\{Z(t); t \in M\}$ having continuous paths and preassigned one dimensional distributions. We always take the state space X for the process to be a metric space and the parameter set M to be a compact topological space. An obvious necessary condition is that the preassigned distributions for the individual $Z(t)$ vary continuously with t. This condition is also sufficient [1], if X is complete metric and M is zero dimensional, the Cantor set for example. If M is anything else, for example an interval on the real line, the simple necessary condition is no longer sufficient as one easily sees, and further conditions on X and on the desired distributions for the $Z(t)$ are needed. We will give here a theorem which treats the case in which M is arbitrary.

First we must introduce some notation and a precise statement of the problem, which also makes it look more like the sort of thing one ordinarily considers.

If Y is a topological space, let $P(Y)$ denote the set of all probability measures on the Borel sets of Y. Let $C(Y)$ denote the continuous bounded real valued functions on Y. We give $C(Y)$ the uniform topology and $P(Y)$ the topology generated by the functions $\mu \to \int f d\mu, f \in C(Y)$. Given a measure space $(\Omega, \mathscr{F}, \mu)$ and a mapping φ from Ω to Y which is measurable relative to $\mathscr{F}$ and the Borel sets of Y let $\varphi\mu$ denote the measure on the Borel sets of Y defined by $\varphi\mu(A) = \mu(\varphi^{-1}(A))$. From now on let M and X be compact metric spaces and let $C(M, X)$ denote the continuous functions from M to X under the uniform topology. Each t in M defines, by evaluation at t, a continuous function from $C(M, X)$ to X. We denote this mapping simply by t, so that $tf = f(t)$ for $f \in C(M, X)$. Let μ be an element of $P(C(M, X))$; then in the notation we have just introduced, $t\mu$ defines an element of $P(X)$, namely, $t\mu(A) = \mu(\{f : f(t) \in A\})$. Moreover the mapping $t \to t\mu$ from M to $P(X)$ is continuous.

In this paper we shall consider the converse construction; that is, given a continuous function T from M to $P(X)$, when is there a measure μ in $P(C(M, X))$ such that $t\mu = T(t)$ for all $t \in M$. Note that any such T defines, via the formula $(T^*f)(t) = \int f dT(t)$, a continuous linear mapping from $C(X)$ to $C(M)$ such that $T^*(1) = 1 = \|T^*\|$; any such mapping T^* arises in this way. The mappings of