RANDOMNESS AND EXTRAPOLATION

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1. Introduction

This article proposes a theory of sequential observation as a basis for a definition of random sequences—which is more general than the approaches inspired by the intuitive situations of gambling and sequential testing. It investigates implications of the constructivist thesis which equates sequential observation and extrapolation, in the case of repeated independent random experiments.

As shown in Section 4 this leads to a definition of the concept of "infinite random sequence" considerably narrower than those proposed by Martin-Löf [10] and Schnorr [21] (a discussion of these approaches can be found in Section 3). There exist sequences random in the sense of Martin-Löf and generated by finite rules (of the class $\Sigma_2 \cap \Pi_2$), revealing the incompatibility of these notions and intuition.

The approach of this article will be guided by the intuitive notion of random phenomena as collections of finite samples which will, on the average, be ultimately observed in sequential experiments. The corresponding class of random sequences does not show pathologies of the type indicated.

In Section 5 "on the average" is interpreted as "with high probability" rather than "with probability 1" (as before), and distribution limit theorems (invariance principles) are stated yielding the probability levels of certain sequentially observable events related to almost sure convergence theorems.

Let $x_1 x_2 \cdots$ be a machine generated sequence subject to sequential observation, information about the computing mechanism not being available. After a large number of observations $x_1 x_2 \cdots x_n$ have been taken it is possible to reconstruct the generating rule from the data; this is equivalent to an extrapolation of $x_1 x_2 \cdots x_n$. The number n being unknown, however, all one can say is that an extrapolation is possible ultimately.

In contrast to the above, assume now that $x_1 x_2 \cdots$ is generated by a random experiment (say, coin tossing). Then, despite considerable regularities that might occur in the first outcomes, the observer will find himself unable to extrapolate, the complexity of a random sequence being unattained by any extrapolation.

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