GENERALIZED INVERSE OF A MATRIX AND ITS APPLICATIONS

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1. Introduction

The concept of an inverse of a singular matrix seems to have been first introduced by Moore [1], [2] in 1920. Extensions of these ideas to general operators have been made by Tseng [3], [4], [5], but no systematic study of the subject was made until 1955 when Penrose [6], [7], unaware of the earlier work, redefined the Moore inverse in a slightly different way. About the same time one of the authors, Rao [8], gave a method of computing what is called a pseudoinverse of a singular matrix, and applied it to solve normal equations with a singular matrix in the least squares theory and to express the variances of estimators. The pseudoinverse defined by Rao did not satisfy all the restrictions imposed by Moore and Penrose. It was therefore different from the Moore–Penrose inverse, but was useful in providing a general theory of least squares estimation without any restriction on the rank of the observational equations. In a later paper, Rao [9] showed that an inverse with a much weaker definition than that of Moore and Penrose is sufficient in dealing with problems of linear equations. Such an inverse was called a generalized inverse (g inverse) and its applications were considered by Rao in [10], [11], [12], [13], and [14].

Some of the principal contributors to the subject since 1955 are Greville [15], Bjerhammer [16], [17], [18], Ben-Israel and Charnes [19], Chipman [20], [21], Chipman and Rao [22], and Scroggs and Odell [23]. Bose [24] mentions the use of g inverse in his lecture notes, "Analysis of Variance" [24]. Bott and Duffin [25] defined what is called a constrained inverse of a square matrix, which is different from a g inverse and is useful in some applications. Chernoff [26] considered an inverse of a singular nonnegative definite (n.n.d.) matrix, which is also not a g inverse but is useful in discussing some estimation problems.

The g inverse satisfying the weaker definition given by Rao [9] is not unique and thus presents an interesting study in matrix algebra. In a publication in 1967 [27], Rao showed how a variety of g inverses could be constructed to suit different purposes and presented a classification of g inverses. The work was later pursued by Mitra [28], [29], who introduced some new classes of g inverses, and Mitra and Bhimasankaram [30], [31]. Further applications of g inverses were considered in a series of papers, Mitra and Rao [32], [33], [34], and Rao [35].