ASYMPTOTICALLY DISTRIBUTION FREE STATISTICS SIMILAR TO STUDENT'S t

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1. A class of statistics

1.1. Let X be a random variable with a continuous distribution function $F(x) = P\{X \leqq x\}$, $X_1, X_2, \cdots, X_n$ a random sample of X, and $X_{(1)} \leqq X_{(2)} \leqq \cdots X_{(n)}$ the corresponding ordered sample. For given $0 < \gamma < 1$, we consider the γ quantile of X

$$\mu_\gamma = F^{(-1)}(\gamma) \tag{1.1.1}$$

and the corresponding sample quantile

$$V_\gamma = X_{(k)}, \tag{1.1.2}$$

where

$$k = [\gamma n] + 1. \tag{1.1.3}$$

We now consider the statistic

$$S_\gamma = \frac{V_\gamma - \mu_\gamma}{X_{(k+r_2)} - X_{(k-r_1)}}, \tag{1.1.4}$$

where r_1, r_2 are integers such that $0 < r_1 < k$, $0 < r_2 \leqq n - k$.

1.2. The statistics of the form (1.1.4) have a structure somewhat similar to Student's t: the numerator is the difference between an estimate of a location parameter (sample quantile V_γ) and that location parameter (population quantile μ_γ), while the denominator is an estimate (sample interquantile range) of a scale parameter (population interquantile range). A more pertinent analogy with the t statistic is this: the statistic S_γ is invariant under linear transformations and hence, if the distribution function $F(\cdot)$ is given, S_γ has for fixed γ, n, r_1, r_2 a probability distribution independent of location and scale parameters; that is the same for all random variables with distribution functions $F\big((x - a)/b\big)$ with arbitrary real a and positive b. One could, therefore, choose, for example, $F(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/2}\, dy$ and tabulate the probability distributions of S_γ for practically meaningful values of γ, n, r_1, r_2; these probability distributions

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