ITERATED LOGARITHM ANALOGUES FOR SAMPLE QUANTILES WHEN $p_n \downarrow 0$

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1. Introduction

This paper is concerned with behavior of the Law of Iterated Logarithm (LIL) type for sample p_n-tiles, $p_n > 0$, when $p_n \downarrow 0$. The results are all stated for uniformly distributed random variables, from which they may easily be translated into results for general laws.

Let $X_1, X_2, \cdots$ be independent identically distributed random variables, uniformly distributed on $[0, 1]$. Let $T_n(x) = \{\text{number of } X_i \leqq x, 1 \leqq i \leqq n\}$, so that $n^{-1}T_n$ is the right continuous *sample distribution function* based on $X_1, X_2, \cdots, X_n$. Define the *sample p_n-tile* $Z_n(p_n)$ as $\min\{z: T_n(z) \geqq np_n\}$. This makes $Z_n(p_n) = np_n$-th order statistic when np_n is a fixed integer. (When $np_n \to \infty$ our results do not depend on the choice of definition of $Z_n(p_n)$ in cases of ambiguity.)

The earliest nontrivial result in this area, due to Baxter [2], is that, for any positive constant c,

$$\limsup_n T_n(c/n) \log\log\log n (\log\log n)^{-1} = 1, \qquad \text{wp } 1. \tag{1.1}$$

On the other hand, it is trivial (and a consequence of Theorem 2 herein, with $k = 1$) that

$$\liminf_n T_n(c/n) = 0, \qquad \text{wp } 1. \tag{1.2}$$

We thus no longer have the symmetry in asymptotic behavior of positive and negative deviations of $T_n(\pi_n) - ET_n(\pi_n)$ that prevails when π_n is constant; indeed, why should we, when $nT_n(c/n)$ is asymptotically Poisson rather than normal?

This difference in behavior means we will have to state results for the two directions of oscillations separately, and (since the analogue of (1.2) will not always be so simple to state) dictates a choice of nomenclature which we had best introduce at the outset: to eliminate possible confusion with reference to the two *directions* of oscillation, we drop the usual "upper or lower class" LIL terms completely, replacing these by "outer or inner class" for sequences $\{f_n\}$ beyond which $T_n(\pi_n)$ moves (in a direction away from $ET_n(\pi_n)$) finitely or infinitely often

Research carried out under ONR contract Nonr 401 (50) and NSF Grant GP 9297.