ON THE NORMAL APPROXIMATION FOR A CERTAIN CLASS OF STATISTICS

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1. Summary and introduction

We consider the class $C(\alpha)$ of asymptotic tests proposed by Neyman [6]. The term of order $n^{-1/2}$ in the normal approximation for the distributions of the test statistics is obtained. Moreover several theorems on conditional distributions are proved. They are used in deriving the main result but they also seem to be of independent interest.

Neyman [6] proposed a class of asymptotic tests called $C(\alpha)$ for the following statistical problem. Let a random variable (r.v.) X have a distribution depending on parameters $\theta = (\theta_1, \cdots, \theta_s)$ and ξ which take their values in open sets $\Theta \subset R^s$ and $\Xi \subset R^1$ respectively. (We denote by R^s, $s = 1, 2, \cdots$, the space of real row vectors $x = (x_1, \cdots, x_s)$ with the Euclidean norm $\|x\| = (xx')^{1/2}$, a prime denoting the transposition.) The hypothesis $H: \xi = \xi_0$, where $\xi_0 \in \Xi$ is a specified value, is to be tested on the basis of n independent observations $X_1, \cdots, X_n$ of the r.v. (In the sequel, we put $\xi_0 = 0$.) The distribution of X is assumed to have a density $f(x; \theta, \xi)$ (with respect to an appropriate measure) which satisfies certain regularity conditions. The $C(\alpha)$ tests are constructed as follows. Let a function $g(x, \theta)$ be such that

$$E_{\theta, 0}\, g(X, \theta) \equiv 0, \qquad E_{\theta, 0}\, g^2(X, \theta) = \sigma^2 < \infty, \qquad \theta \in \Theta. \tag{1.1}$$

(The first assumption can always be satisfied by considering $g(x, \theta) - E_{\theta, 0}\, g(X, \theta)$ instead of $g(x, \theta)$.) Form the function

$$Z_n(\theta) = \frac{1}{\sigma(\theta)\sqrt{n}} \sum_{i=1}^{n} g(X_i, \theta) \tag{1.2}$$

and let $\hat{\theta}_n$ be a locally root n consistent estimator of θ (which means that $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is bounded in probability where θ_0 is the true value of θ; for a precise definition see [6]). It was shown in [6] that $Z_n(\hat{\theta}_n)$ is asymptotically $(0, 1)$ normally distributed whatever be the true value of θ if and only if $g(x, \theta)$ is orthogonal to the logarithmic derivatives

$$h_j(x, \theta) = \frac{\partial}{\partial \theta_j} \log f(x; \theta, 0), \qquad j = 1, \cdots, s, \tag{1.3}$$