SEQUENTIAL RANK TESTS—ONE SAMPLE CASE

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1. Introduction

Let $X_1, X_2, \cdots$ be a sequence of independent randon variables, identically distributed according to the continuous c.d.f. F. The null hypothesis is H_0: $F(-x) = 1 - F(x)$, $0 \leqq x < +\infty$; that is, the random variables are symmetrically distributed about zero. Sequential tests of this hypothesis which are based on the signs and ranks of the X_i are studied in this paper.

Sequential rank tests should be particularly useful in medical clinical trials. The one sample case arises naturally when patients are paired for similarity of influential physical traits, and are randomly assigned to one of two possible treatments so that each treatment is given to one member of each pair. The variable X_i is the difference between the treatment effects measured on the ith patient pair. Sequential binomial trials have been valuable in this context and will continue to be so. Rank tests, however, can take advantage of quantitative (nondichotomous) information in each treatment comparison while at the same time making only minimal assumptions about the form of the distribution.

In 1969 Weed, Bradley, and Govindarajulu [4] proposed a sequential likelihood ratio test for this problem. Let $G(x) = P\{|X| < x | X < 0\}$ and $H(x) = P\{X < x | X > 0\}$. They considered the family of distributions whose left and right tails are related by $1 - H(x) = (1 - G(x))^A, A > 0$, and $F(0) = A/(1 + A)$. For this family the null hypothesis becomes $H_0 : A = 1$, and an alternative hypothesis is $H_1 : A = B$ where B is a specified constant. The likelihood for the signs and rank order of the absolute values of $X_1, \cdots, X_n$ is

$$(1)\qquad \binom{n}{n^-}\left(\frac{A}{1+A}\right)^{n^-}\left(\frac{1}{1+A}\right)^{n^+} n^-!\, n^+! \int_{0<x_1<\cdots<x_n}\cdots\int \prod_{i=1}^{n} \{dG(x_i)\}^{\delta_i}\{dH(x_i)\}^{1-\delta_i},$$

where $\delta_i = 1$ if the X_i with the ith smallest absolute value is negative, $\delta_i = 0$ if it is positive. For $H_0 : A = 1$ and $H_1 : A = B$, the likelihood ratio simplifies to

$$(2)\qquad LR_n = \frac{(B/1 + B)^n 2^n n!}{\prod_{i=1}^{n} [n_i^- + Bn_i^+]},$$

where n_i^- is the number of X_j such that $X_j < 0$ and $|X_j| \geqq |X_i|$ and n_i^+ is the