ON INEQUALITIES OF CRAMÉR-RAO TYPE AND ADMISSIBILITY PROOFS

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1. Introduction and Summary

This paper is a discussion of the Hodges-Lehmann [7] method of proving admissibility for quadratic loss.

Section 2 compares the inequality $EU^2 \geqq (EU)^2$ with the best Schwarz inequality $EU^2 \geqq (EU)^2 + \{\text{Cov}(U, V)\}^2/\text{Var } V$ obtainable using linear functions of U and V, and considers invariance properties of these inequalities.

In Section 3, for a random variable X with possible distributions indexed by θ, we define a Cramér-Rao type inequality as one giving a lower bound on $\text{Var } T(X)$ in terms of $ET(X)$. Theorem 2 shows that for the best Schwarz inequality $\text{Var } T \geqq \{\text{Cov}(T, V)\}^2/\text{Var } V$ using $V = V(X, \theta)$ to be of Cramér-Rao type, it is necessary that V depend on X only through a minimal sufficient statistic; this condition is also sufficient when there is a sufficient statistic with a complete family of possible distributions. In this case of completeness, it follows that the Cramér-Rao and Bhattacharyya inequalities require no regularity conditions beyond existence and nonconstancy of the derivatives involved.

Section 4 describes the Hodges-Lehmann method of proving an estimator T^* admissible for quadratic loss. In this method, the inequality showing that T makes T^* inadmissible is relaxed using the Cramér-Rao type inequality $\text{Var}(T - T^*) \geqq \{\text{Cov}(T - T^*, T^*)\}^2/\text{Var } T^*$, and the relaxed inequality is shown to have no nontrivial solutions. In all examples known to us, this use of a Cramér-Rao type inequality can be replaced by a use of the weaker result $\text{Var}(T - T^*) \geqq 0$; we suppose there are examples in which this cannot be done, but we have no such example.

Section 5 consists of several examples illustrating this method of proving admissibility.

2. Schwarz's inequality

For real valued random variables U and V, Schwarz's inequality

$$\{EU^2\}\{EV^2\} \geqq \{EUV\}^2 \tag{2.1}$$

means that if EU^2 and EV^2 both exist, then EUV also exists and its square does not exceed their product.

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