BOUNDS ON INTERVAL PROBABILITIES FOR RESTRICTED FAMILIES OF DISTRIBUTIONS

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1. Introduction

A number of improvements of the classical Chebyshev inequalities are known that depend on various restrictions in addition to moment conditions. Most of these results provide bounds on the distribution function $P\{X \leq t\}$. In this paper, we consider bounds on $P\{s < X \leq t\}$, $P\{s < X \leq t|X \leq t\}$ and on $P\{s < X \leq t|X > s\}$. Bounds are also obtained on densities and hazard rates. These bounds are obtained under a variety of restrictions, but a unified method is used which yields all results as special cases of a single theorem.

The restrictions we impose yield quite striking improvements over what is obtainable with moment conditions alone. Furthermore, at least some of the conditions arise in practice and can be verified under the proper circumstances by physical considerations. In all cases we assume that $P\{X \geq 0\} = 1$.

From a historical viewpoint, a natural condition to consider is that $1 - F(x) = P\{X > x\}$ is convex on $[0, \infty)$. Bounds in this case were obtained by Gauss; a number of extensions and related results have been summarized by Fréchet [7]. Such bounds are often stated as inequalities on $P\{|Y - m| > x\}$ where Y is unimodal with mode m. Of course this implies that $X = |Y - m|$ satisfies $P\{X \geq 0\} = 1$ and $P\{X > x\}$ is convex.

In recent papers (Barlow and Marshall [2], [3]) we considered the condition that the distribution has a monotone hazard rate. If F has a density f, the ratio $q(x) = f(x)/[1 - F(x)]$ is defined for $F(x) < 1$ and is called the *hazard rate*, or sometimes the failure rate or force of mortality. Whether or not F has a density, F is said to have an increasing (decreasing) hazard rate—denoted IHR (DHR)—if $\log [1 - F(x)]$ is concave where finite (convex on $[0, \infty)$). It is easily seen that in case q exists, this property is equivalent to q increasing (decreasing). If F is a life distribution, $q(x)\, dx$ can be interpreted as the conditional probability of death in $[x, x + dx]$ given that death has not occurred before x. Because of

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