EXISTENCE OF BOUNDED INVARIANT MEASURES IN ERGODIC THEORY

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1. Introduction

We present a survey of some of the recent work done on the problem of existence of bounded invariant measure for positive contractions defined on L^1-spaces.

2. Preliminaries

1. *Positive linear forms on L^∞-spaces.* Let $(E, \mathfrak{F}, \mu)$ be a fixed measure space (with μ σ-finite). Sets in $\mathfrak{F}$ and real measurable functions defined on $(E, \mathfrak{F})$ will always be considered up to μ-equivalence; hence, all equalities or inequalities between measurable sets or functions are to be taken in the almost sure sense with respect to μ.

We will denote by f, g (with or without subscripts) elements of the Banach space $L^1(E, \mathfrak{F}, \mu)$ and by h elements of the Banach space $L^\infty = L^\infty(E, \mathfrak{F}, \mu)$. The space L^∞ is the strong dual of L^1 for the bilinear form: $\langle f, h\rangle = \int_E fh\, d\mu$. Consideration of the strong dual of L^∞, in which L^1 is isometrically imbedded, has often been helpful in analysis. We here recall the following lemma from the theory of vectorial lattices, of which we sketch a proof out of completeness.

LEMMA 1. *Let λ be a positive linear form defined on L^∞; that is, let $\lambda \in (L^\infty)'_+$. Then there exists a largest element g in L^1_+ such that the form induced by it on L^∞ verifies $g \leq \lambda$. Moreover, the complement $G = \{g = 0\}$ of the support of g is the largest set in $\mathfrak{F}$ (up to equivalence) for which there exists an $h \in L^\infty_+$ such that $h > 0$ on G and $\lambda(h) = 0$; in particular, the following equivalences hold:*

(a) $g > 0$ *a.s.* $\Rightarrow \lambda(h) > 0$ *for every* $h \in L^\infty_+$, $h \neq 0$.

(b) $g = 0$ *a.s.* $\Rightarrow \lambda(h) = 0$ *for at least one* $h \in L^\infty$ *such that* $h > 0$ *a.s.*

PROOF. The class $\Lambda = \{f\colon f \in L^1_+, f \leq \lambda \text{ on } L^\infty_+\}$ is easily seen to be closed under least upper bounds and increasing limits; hence, $g = \sup \Lambda$ belongs to Λ, and is thus the largest element of Λ.

Given two linear forms ν_1, ν_2 on L^∞, it is known and easily checked that the formula $\nu(h) = \inf \{[\nu_1(u) + \nu_2(h - u)]; 0 \leq u \leq h\}$ where $h \in L^\infty_+$, defines on L^∞_+ a linear form ν on L^∞, which is the g.l.b. of ν_1 and ν_2. Now it follows from the