INVARIANT MEASURES ON PRODUCT SPACES

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1. Introduction

Let G be a locally compact group (which we always assume to satisfy the second axiom of countability), and let X be a standard Borel space on which G acts as a Borel transformation group (see [7], p. 628). That is, we have a homomorphism of G into the group of Borel automorphisms of X such that if $x \to g \cdot x$ is the automorphism corresponding to $g \in G$, then $(x, g) \to g \cdot x$ is a Borel function from $G \times X$ into X where G is endowed with the σ-field of sets generated by the open sets, and $G \times X$ is given the product σ-field. Further, let μ be a σ-finite measure on X (all measures henceforth will be understood to be σ-finite) which is quasi-invariant under G; that is, for every $g \in G$, $g \cdot \mu$ and μ are equivalent in the sense of mutual absolute continuity. (Here $g \cdot \mu$ is the transform of μ by g defined by $(g \cdot \mu)(\sigma) = \mu(g^{-1} \cdot \sigma)$ for Borel sets σ of X.) One says that μ is ergodic under G if for every Borel σ in G such that $\mu(\sigma \Delta g \cdot \sigma) = 0$ for all $g \in G$, we have $\mu(\sigma) = 0$ or $\mu(X - \sigma) = 0$. It is clear that these two properties of μ depend not on μ, but only on the equivalence class $C(\mu)$ of μ. We shall say, following [8], that $C(\mu)$ is a quasi-orbit of G if μ is quasi-invariant and ergodic. Note that each orbit of G on X carries a unique equivalence class $C(\mu)$ of such measures (see [8], p. 295). One calls these classes transitive quasi-orbits or simply orbits.

We shall say that a measure ν is invariant if $g \cdot \nu = \nu$ for all $g \in G$. In this note we are going to discuss a special case of the following circle of questions: given a quasi-orbit $C(\mu)$ on X, when does it contain an invariant measure ν or more specifically an invariant ν with specified properties? We note that if $\nu \in C(\mu)$ is invariant, it is unique up to multiplication by positive scalars. This is an immediate consequence of ergodicity. Furthermore, any $\lambda \in C(\mu)$ is either atomic (consists of point masses) or nonatomic (no point masses).

The systems $(G, X, C(\mu))$ which we will discuss will be such that G is countable and acts freely on X in the sense that $\{x \colon g \cdot x = x \text{ for some } g \neq e\}$ is a μ-null set. When these conditions are satisfied, von Neumann in [10] has shown how to construct a certain factor von Neumann algebra associated with $(G, X, C(\mu))$. The type of this factor is determined by the measure theoretic properties of $C(\mu)$ discussed in the previous paragraph (see [10], theorem IX). One can show

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