ERGODIC THEORY OF SHIFT TRANSFORMATIONS

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1. Introduction

The main purpose of this note is to discuss the ergodic properties of a certain class of strictly ergodic dynamical systems which appear as subsystems of the shift dynamical system defined on the power space $X = A^Z$, where Z is the set of all integers. We discuss only the cases when the base space A is a finite set. We are particularly interested in two examples of strictly ergodic dynamical systems which are constructed by using certain number-theoretic functions. Among other things it will be shown that there exist a continuum number of strictly ergodic dynamical systems, no two of which are spectrally isomorphic.

2. Strictly ergodic dynamical systems

Let $X = \{x\}$ be a nonempty compact metrizable space, and let φ be a homeomorphism of X onto itself. The pair (X, φ) is called a *dynamical system.* A subset X_0 of X is said to be φ-invariant if $\varphi(X_0) = X_0$. If X_0 is a nonempty closed φ-invariant subset of X, then (X_0, φ) may be considered as a dynamical system, and is called a *dynamical subsystem* of (X, φ). A dynamical system (X, φ) is said to be *minimal* if there is no dynamical subsystem of (X, φ) except (X, φ) itself, that is if there is no nonempty closed φ-invariant subset of X except X itself.

Let

$$(1) \qquad Z = \{n | n = 0, \pm 1, \pm 2, \cdots\}$$

be the set of all integers. For any point $x_0 \in X$, the set

$$(2) \qquad \text{Orb}\,(x_0) = \{\varphi^n(x) | n \in Z\}$$

is called the *orbit* of x_0, and its closure $\overline{\text{Orb}}\,(x_0)$ is called the *orbit closure* of x_0. Obviously, $\overline{\text{Orb}}\,(x_0)$ is a closed φ-invariant subset of X, and hence $(\overline{\text{Orb}}\,(x_0), \varphi)$ is a dynamical subsystem of (X, φ). It is clear that a dynamical system (X, φ) is minimal if and only if Orb (x_0) is dense in X for any $x_0 \in X$.

Let $\mathcal{B} = \{B\}$ be the σ-field of all Borel subsets B of X. It was proved by *N.* Kryloff and N. Bogoliouboff [6] that, for any dynamical system (X, φ), there exists a normalized, countably additive, nonnegative measure μ defined on $\mathcal{B}$ which is invariant under φ; that is, $\mu(\varphi(B)) = \mu(B)$ for any $B \in \mathcal{B}$. Such a measure μ is not necessarily unique. A dynamical system (X, φ) is said to be *uniquely ergodic*