ON POINCARÉ'S RECURRENCE THEOREM

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1. Introduction

Let T be a continuous mapping of a Polish space Ω into itself. Assume that some randomness is introduced into Ω by a normalized measure m. Then we may distinguish between macroscopic and microscopic properties of the system; the macroscopic properties concern the behavior of m, whereas the microscopic properties concern the behavior of the individual points of Ω, under the action of T.

Poincaré's classical recurrence theorem (see, for example, Jacobs [6], p. 49, ff.) says, roughly speaking, that macroscopic stationarity implies microscopic recurrence. Here the statement concerning the behavior of the system is weakened in passing from the macroscopic hypothesis to the microscopic conclusion of the theorem. Imagine a sequence of systems, each governing by its microscopic behavior (of one of its points) the macroscopic behavior of the subsequent one. If in the first system T is the identity mapping, then the second one will be macroscopically stationary, hence (according to Poincaré) the third system will be 'macroscopically recurrent.' Poincaré's theorem does not say what the third system will do microscopically. Theorem 3.1 of the present paper asserts that also the third system will be microscopically recurrent, and that recurrence will never get lost throughout the whole dynasty of systems linked in the indicated way: recurrence is a hereditary property.

Of course we first have to make precise the concepts of macroscopic and microscopic recurrence. Section 2 of this paper is devoted to the definition and discussion of recurrence of points and measures under the action of T; for measures, weak topology is adopted, and examples and easy constructions are exhibited.

In section 3 the new recurrence theorem (theorem 3.1) is stated. We prove it in two different ways. I had proved the theorem for mixing measures (section 5) but not for the general case, when I told the problem to V. Strassen (Goettingen). After 24 hours we met again, each having produced a proof for the general case. Strassen's proof is by far the simpler one, employing the same ideas which are used for the classical Poincaré theorem; my proof needs some preparation and uses the ideas of M. Kac [7] (see also Jacobs [6], p. 55, ff.); both proofs are given here.

Section 4 gives the lattice properties of the system of all measures which are