ROOTS OF THE ONE-SIDED N-SHIFT

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1. Introduction and summary

In his booklet on ergodic theory [1] Halmos raises the question of the existence of p-th roots of measure-preserving transformations, and more specifically the question of the existence of p-th roots of the N-shifts (see problem 4 on page 97). On page 56 of the same book he indicates that if $N = k^2$, then the N-shift has a square root. Clearly, essentially the same argument shows that if $N = k^p$, then the N-shift has a p-th root.

The main purpose of this paper is to show that the one-sided N-shift has a p-th root if and only if $N = k^p$ for some positive integer k. The problem of the existence of roots seems to be more difficult for the bilateral N-shift than for the one-sided N-shift. At least our methods involve the many-to-one nature of the one-sided N-shift and its roots, and cannot be used on the bilateral shifts.

2. Notation

The following symbols will be used:

- N is a positive integer;
- $\Omega = \{\omega = (\omega_1, \omega_2, \cdots) | \omega_i \in \{0, 1, \cdots, N-1\} \text{ for all } i\}$;
- Σ is the smallest σ-field containing all sets of the form $\{\omega | \omega_i = k\}$;
- P is a probability measure on (Ω, Σ) defined so that the sequence $\{\omega_k\}$ of coordinate projection random variables is an independent sequence, and so that $P\{\omega | \omega_i = k\} = 1/N$ for $k = 0, 1, \cdots, N-1$ and all i;
- T is the one-sided N-shift defined by $T(\omega_1, \omega_2, \cdots) = (\omega_2, \omega_3, \cdots)$;
- Σ^0 is the subcollection of 2^Ω consisting of all subsets of sets (in Σ) of measure zero;
- $\Sigma^* = \{E_1 + E_2 | E_1 \in \Sigma \text{ and } E_2 \in \Sigma^0\}$. This is a σ-field;
- P^* is the completion of P to Σ^*;
- $\omega + j/N = (\omega_1', \omega_2, \cdots)$ where $\omega = (\omega_1, \omega_2, \cdots)$, $0 \le \omega_1' \le N-1$, and $\omega_1' = \omega_1 + j \pmod{N}$.

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