RANDOM MEASURE PRESERVING TRANSFORMATIONS

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1. Introduction

It is the purpose of this note to show that it is impossible to define a probability measure on the group $\mathcal{G}$ of invertible measure-preserving transformations from the unit interval onto itself, if it is demanded that the measure on $\mathcal{G}$ obey two fairly "natural" conditions. One of these is an invariance condition on the measure, and the other asserts that certain distinguished subsets of $\mathcal{G}$ are measurable.

One reason for trying to construct such a probability measure is the following: the group $\mathcal{G}$ has been topologized in at least two different ways (see Halmos [3]); in one of those topologies (the "weak" topology) it has been proved that the set $\mathcal{E}$ of ergodic transformations (and in fact, the set $\mathcal{W}$ of weakly mixing transformations) is of the second category, and the set $\mathcal{S}$ of strongly mixing transformations is of the first category (see [3], p. 77 ff.). Corresponding to this information about the "topological size" of $\mathcal{E}$, $\mathcal{W}$, and $\mathcal{S}$, it would have been natural to seek information about the measures of these (and possibly other) subsets of $\mathcal{G}$. One could have hoped, for example, that "almost every transformation is ergodic." However, one needs first to have an appropriate measure on $\mathcal{G}$.

Another motivation comes from game theory. One of the characterizations of the Shapley value [4] of a cooperative n-person game involves a random ordering of the players. Recently games in which the player set may be a (possibly atomless) measure space have attracted attention, in part because of their applications to economics and politics. (For a comprehensive list of references, see Debreu [2].) One approach to defining the Shapley value for such games would involve the notion of a "random ordering" of the measure space of players. Replacing "ordering" with "measure-preserving transformation," leads to the question that we have answered (negatively) in this note.

The theorem of this paper provides additional evidence of the comparative intractability of function spaces when viewed from the measure-theoretic rather than from the topological viewpoint (compare with [1]).

A precise statement of the theorem is given in section 2, and it is proved in

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