RENEWAL THEOREMS FOR MARKOV CHAINS

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1. Introduction

This study originated in an attempt to understand the meaning of two parameters, the mean and the variance, which appear in the limit theorems concerning sums of identically distributed independent random variables. The law of large numbers, when written in the usual form ($\lim S_n/n = \mu$) does not immediately suggest a natural generalization to Markov chains; the reason is that a Markov chain takes its values in an abstract state space so that not only the limit μ, but also the ratios which tend to μ have to be reinterpreted in a meaningful fashion. Therefore, we proceed to the renewal theorem in the form proved by Erdös, Feller, and Pollard (cf. [3], p. 286) and Chung and Wolfowitz [2], and formulate it in a manner which readily suggests a natural generalization.

Consider a random walk (spatially homogeneous Markov chain) x_n on the integers with transition function $P(x, y)$ defined for arbitrary pairs of integers x, y. Suppose that it satisfies

$$P(x, y) \geq 0, \qquad \sum_{y=-\infty}^{\infty} P(x, y) = 1, \tag{1.1}$$

$$P(x, y) = P(x + z, y + z) \qquad \text{for all} \quad z, \tag{1.2}$$

$$\sum_{x=-\infty}^{\infty} |x|\, P(0, x) < \infty, \qquad \sum_{x=-\infty}^{\infty} xP(0, x) = \mu > 0, \tag{1.3}$$

$$\sum_{y=-\infty}^{\infty} P(x, y)f(y) = f(x) \quad \text{and} \quad |f(x)| \leq 1 \Rightarrow f(x) = \text{constant}. \tag{1.4}$$

The last condition (1.4) is well known to be equivalent to the usual aperiodicity requirement that the support of $P(0, x)$ is not contained in a proper subgroup of the integers (cf. [4], p. 276). This Markov chain is transient in view of condition (1.3), ([4], p. 33), and the renewal theorem is a simple statement concerning the asymptotic behavior of the Green function $G(x, y)$ defined by

$$\begin{aligned} G(x, y) &= \sum_{n=0}^{\infty} P_n(x, y), \\ P_0(x, y) &= \delta(x, y), \qquad\qquad (1.5) \\ P_{n+1}(x, y) &= \sum_{z=-\infty}^{\infty} P_n(x, z)P(z, y), \qquad n \geq 0. \end{aligned}$$

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