SOME PECULIAR SEMI-MARKOV PROCESSES

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SUMMARY. Experience with semi-Markov processes with finite expected *waits* suggests that the behavior of Markov processes is a good guide to understanding the behavior of the more general process. However, examples are given to show that when expected waits are infinite quite surprising behavior is possible. For a two-state aperiodic semi-Markov process the instantaneous state probabilities $P_i(t)$ can have $(C, 1)$-limits but not strict limits; for a three-state (and *irreducible*) process one can have $P_0(t)$ tend to a strict limit as $t \to \infty$ but $P_1(t)$ and $P_2(t)$ not even have $(C, 1)$-limits. For an aperiodic irreducible infinite chain one can have $P_i(t) \to \pi_i > 0$ as $t \to \infty$, for every i, yet $\Sigma \pi_i < 1$.

1. Introduction

Semi-Markov processes were introduced simultaneously by Lévy [3] and by Smith [7], [8]. The constructive definition of Smith, which is valuable so long as only a few states are instantaneous, has been given an elaborate and formal treatment by Pyke [5], [6].

For the present note we shall suppose that we are given

(i) the transition matrix $\|p_{ij}\|$ of an irreducible and recurrent Markov chain of, possibly, infinitely many states;

(ii) a sequence $\{\Omega_i(x)\}$ of proper distribution functions of nonnegative random variables and such that there is at least one i such that $\Omega_i(0+) < 1$.

We imagine the process develops as follows. An initial state, say i_0, is selected, and the process stays in this state for a period of time governed by the distribution function $\Omega_{i_0}(x)$. At the end of this *wait* in the state i_0 the process then selects a fresh state in accordance with the transition matrix $\|p_{ij}\|$; thus, with probability $p_{i_0 i_1}$ the system now moves to state i_1. Having reached state i_1 the system waits there a period of time governed by $\Omega_{i_1}(x)$, and so on. It is assumed that successive waits are independent. Under the assumptions we are presently making (especially the recurrence of $\|p_{ij}\|$) there will be, with probability one, finitely many transitions in any finite time period. To avoid ambiguity we may suppose that at the instant of a transition the system is in the state in which it will next reside for a strictly positive amount of time.

For purposes of discussion, let us suppose (with no loss of generality) that the

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