LIMITING DISTRIBUTIONS FOR BRANCHING PROCESSES

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1. Introduction

Let Z_n, $n = 0, 1, 2, \cdots$, be the number of individuals in the n-th generation of a Galton-Watson branching process with basic distribution $\{p_i, i = 0, 1, \cdots\}$. That is, $\{Z_n\}$ are the random variables of a Markov chain whose states are the nonnegative integers and whose transition probability matrix is defined by

$$(1.1) \qquad p_{ij} = \text{coeff. of } x^j \text{ in } p(x)^i, \qquad \text{where} \quad p(x) = \sum_{\ell=0}^{\infty} p_\ell x^\ell.$$

For background on these processes we refer to the monograph of T. E. Harris [3], and all statements to the effect that some property of branching processes is "well known" are hereby defined to mean that the property in question is discussed there.

The purpose of this paper is to undertake a systematic study of limit distributions for Z_n, where the initial state Z_0 is allowed to tend to infinity with n. Thus it may happen, for certain sequences of numbers a_n, $b_n > 0$, c_n = positive integer ($c_n \to \infty$), that

$$(1.2) \qquad \lim_{n\to\infty} P\left\{\frac{Z_n - a_n}{b_n} \le x \,\middle|\, Z_0 = c_n\right\} = G(x)$$

exists in the usual sense of weak convergence of distribution functions. The basic problem, which is far from being completely solved, is to determine the class of distributions G which can arise in this manner, and the conditions on $\{p_i\}$ and $\{c_n\}$ under which a particular G will appear. If the distribution $\{p_i\}$ has finite variance, these questions can be fully answered with little difficulty; this is done in section 2 below. The case of infinite variance seems much more difficult, however, and only fragmentary results have been achieved so far (section 3).

A closely related problem occurs when (1.2) is strengthened by replacing Z_n by $Z_{[nt]}$ and requiring convergence to a limit (depending on t) for each $t \ge 0$. This is essentially equivalent to asserting the existence of a limiting process, and can occur only in the "critical" case $\mu = 1$. If the convergence occurs without translating the process—that is, if $a_n \equiv 0$—it has been possible to find all of the limiting processes which can arise. This theorem is the subject of section 4, and states that the possible limits form a one-parameter family (apart

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