ON LOCAL AND RATIO LIMIT THEOREMS

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1. Introduction

In this paper we obtain local limit theorems, local limit theorems for large deviations, and ratio limit theorems for multi-dimensional probability measures which may be lattice, nonlattice, or a combination of the two.

2. Statements of results

Let R^d denote the set of d-tuples of real numbers $x = (x^1, \cdots, x^d)$. Let μ denote a probability measure on the Borel subsets of R^d with characteristic function f defined by

$$f(\theta) = \int_{R^d} e^{ix\cdot\theta}\mu(dx), \qquad \theta = (\theta_1, \cdots, \theta_d) \in R^d, \tag{2.1}$$

where $x\cdot\theta = x^1\theta_1 + \cdots + x^d\theta_d$.

We assume that μ is nondegenerate in that it is not supported by any $(d-1)$-dimensional affine subspace of R^d. Then by making a suitable linear transformation on R^d, we can assume that μ is normalized in the following sense (see Spitzer [10], pp. 64–75): there is an integer d_1, $0 \leq d_1 \leq d$, and there are real numbers $\alpha^1, \cdots, \alpha^{d_1}$ such that

$$f(2\pi n_1, \cdots, 2\pi n_{d_1}, 0, \cdots, 0) = \exp(2\pi i(n_1\alpha^1 + \cdots + n_{d_1}\alpha^{d_1})) \tag{2.2}$$

for integral $n_1, \cdots, n_{d_1}$, and $|f(\theta)| < 1$ for all other values of θ. If $d_1 = d$, then μ is lattice and if $d_1 = 0$, then μ is nonlattice.

Let $\mu^{(n)}$ denote the n-fold convolution of μ with itself. It is clear that $\mu^{(n)}$ is supported by

$$D_n = \{x \in R^d | x^k - n\alpha^k \text{ is an integer for } 1 \leq k \leq d_1\}. \tag{2.3}$$

Note that D_n is independent of n if and only if we can take $\alpha^1 = \cdots = \alpha^{d_1} = 0$, and in particular, that $D_n = R^d$ if $d_1 = 0$. The statements below can be simplified somewhat in these cases.

For the $0 \leq h < \infty$ set

$$I_h = \{x \in R^d \mid |x^k| \leq h/2 \text{ for } 1 \leq k \leq d\}, \tag{2.4}$$

and

$$\bar{I}_h = \{x \in R^d | x^k = 0 \text{ for } 1 \leq k \leq d_1 \text{ and } |x^k| \leq h/2 \text{ for } d_1 < k \leq d\}. \tag{2.5}$$

Also set $x + I_h = \{y | y - x \in I_h\}$ and $x + \bar{I}_h = \{y | y - x \in \bar{I}_h\}$.