SOME LOCAL PROPERTIES OF MARKOV PROCESSES

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1. Introduction

In 1953, Lévy [3] proved that for almost all Brownian motion paths $X(t)$ in the Euclidean space R^N of dimension $N \geq 3$,

$$\Lambda_\rho(\{X(\tau)\colon 0 \leq \tau \leq t\}) \leq Kt, \tag{1}$$

where Λ_ρ is the Hausdorff measure in R^N formed with the function $\rho(a) = a^2 \log\log a^{-1}$, and conjectured that

$$\Lambda_\rho(\{X(\tau)\colon 0 \leq \tau \leq t\}) \geq kt \tag{2}$$

with probability one.

Lévy's conjecture was proved in 1961 by Ciesielski and Taylor [2]. The use of a density theorem of Rogers and Taylor [5] enabled them to obtain (2) by proving that with probability one,

$$\limsup_{a \to 0} T(a, t)/\rho(a) = c_N, \tag{3}$$

where

$$T(a, t) = \int_0^t V(X(\tau); a)\, d\tau, \tag{4}$$

$$V(x; a) = 1, \qquad |x| \leq a, \tag{5}$$
$$= 0, \qquad |x| > a,$$

is the sojourn time up to time t of the path inside a sphere of radius a about the initial point $X(0) = 0$. (Actually, the proof of (2) used only the fact that the lim sup in (3) is bounded below with probability one.) The constant c_N is expressed in terms of the zeros of Bessel functions through an eigenvalue problem for Laplace's equation.

In [2], Ciesielski and Taylor conjectured in turn that the result (3) holds also for $N = 2$ if the function ρ is chosen to be $\rho(a) = a^2 \log\log\log a^{-1}$. This was proved in [4], with the implication, as in [2], that the lower bound (2) holds with probability one for planar Brownian motion, with the above choice of ρ. The proper constant for (3) in this case turned out to be $c_2 = \frac{1}{2}$. Finally, Taylor [6] used (3) and related results to extend (1) to the planar case.

The point is that Taylor's work showed that properties (1) and (2) of the

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