ON QUASI-COMPACT PSEUDO-RESOLVENTS

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1. Introduction

Let $\{R_\lambda: \lambda > 0\}$ be a family of continuous linear operators on a Banach space such that

$$\left.\begin{aligned} \|\lambda R_\lambda\| &\leq 1 \\ R_\lambda - R_\mu + (\lambda - \mu)R_\lambda R_\mu &= 0 \end{aligned}\right\} \qquad \text{for} \quad \lambda, \mu > 0. \tag{1}$$

Suppose that, for some $\alpha > 0$, αR_α is quasi-compact; that is, there exist a positive integer m, a linear D with $\|D\| < 1$, and a compact K such that $(\alpha R_\alpha)^m = K + D$. Then we shall show that λR_λ is quasi-compact for *all* $\lambda > 0$. (In the special case where $\{R_\lambda\}$ is the resolvent family of a strongly continuous Markov semigroup on ℓ_1, the result is proved in the accompanying paper by David Williams [1].)

2. A lemma on pseudo-resolvents

LEMMA. *Suppose* $\{S_\lambda: \lambda > 0\}$ *is a family of elements of a Banach algebra with identity* (*of norm* 1) *such that*

$$\left.\begin{aligned} \|\lambda S_\lambda\| &\leq 1 \\ S_\lambda - S_\mu + (\lambda - \mu)S_\lambda S_\mu &= 0 \end{aligned}\right\} \qquad \text{for} \quad \lambda, \mu > 0. \tag{2}$$

Then for all $\lambda, \mu > 0$,

(*i*) *the points* z_λ *in* $\sigma(S_\lambda)$ (*the spectrum of* S_λ) *are precisely those points of the form*

$$z_\lambda = \frac{z_\mu}{1 - (\mu - \lambda)z_\mu} \quad \text{where} \quad z_\mu \in \sigma(S_\mu), \tag{3}$$

and (ii) $\sigma(\mu S_\mu) \subseteq \{z: |z - \tfrac{1}{2}| \leq \tfrac{1}{2}\}$.

PROOF. By repeated substitution of $S_\lambda = (1 + (\mu - \lambda)S_\lambda)S_\mu$ into itself, we get

$$S_\lambda = \sum_{n=0}^{N} (\mu - \lambda)^n S_\mu^{n+1} + (\mu - \lambda)^{N+1} S_\lambda S_\mu^{N+1}. \tag{4}$$

Therefore, provided that $(\mu - \lambda)^N \|S_\mu^N\| \to 0$ as $N \to \infty$,

$$S_\lambda = \sum_{n=0}^{\infty} (\mu - \lambda)^n S_\mu^{n+1}. \tag{5}$$

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