# UNIFORM ERGODICITY IN MARKOV CHAINS

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## 1. Introduction

Let $\ell$ denote the complex Banach space of vectors $x = (x_1, x_2, \cdots)$ with $\|x\| \equiv \sum |x_i| < \infty$. It is known (see, for example, Hille and Phillips [3], section 23.12) that the equation

$$(1) \qquad (P_t x)_j = \sum_i x_i p_{i,j}(t), \qquad (j = 1, 2, \cdots; t \geq 0)$$

sets up a biunique correspondence between those Markov transition matrix functions $\{p_{i,j}(t): t \geq 0; i, j = 1, 2, \cdots\}$ for which

$$(2) \qquad \lim_{t \downarrow 0} p_{j,j}(t) = 1, \qquad (j = 1, 2, \cdots)$$

('standard' transition matrix functions) and strongly continuous semigroups $\{P_t: t \geq 0\}$ of positive transition operators on $\ell$. Let $\Omega$ denote the infinitesimal generator of such a semigroup, and let $\{R_\lambda: \lambda > 0\}$ denote the resolvent family of $\Omega$ so that

$$(3) \qquad R_\lambda x = \int_0^\infty \exp(-\lambda t) P_t x \, dt, \qquad (\lambda > 0)$$

(see Hille and Phillips ([3], chapter XI)).

From the resolvent equation, $R_\lambda - R_\mu + (\lambda - \mu) R_\lambda R_\mu = 0$, it follows that if $R_\lambda$ is compact for some $\lambda > 0$, then $R_\mu$ is compact for every $\mu > 0$. Many of the special Markov chains which have been studied analytically have been shown to possess compact resolvents; for example, the chains $K1$ and $K2$ analyzed by Kendall and Reuter ([5], section 6) and the chain constructed by Kendall in reference [4] (see [5], (5)). Perhaps the best reason for interest in the condition '$R_\lambda$ compact' is that, for a wide class of semigroups including Markov semigroups on $\ell$, it is equivalent to *reflexivity* in the sense of Phillips [9]. This result follows from theorem II of Kendall's paper [8].

An operator $T$ is called *quasi-compact* if there exist a positive integer $m$ and a compact operator $C$ such that $\|T^m - C\| < 1$. Kendall and Reuter ([6], theorem 7) have shown that, under general conditions, the statement "*there exists a (compact) projection operator* $\Pi$ *of finite dimensional range such that* $\mu R_\mu \to \Pi$ *in the uniform topology as* $\mu \downarrow 0$" is equivalent to the statement "$\lambda R_\lambda$ *is quasi-compact for some* $\lambda > 0$." (Kendall and Reuter actually only show that

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