SOME PROBLEMS RELATING TO MARKOV GROUPS

JANE M. O. SPEAKMAN
UNIVERSITY OF CAMBRIDGE

1. Introduction

This paper forms a sequel to one by Professor D. G. Kendall [2]. We shall use the notation of that paper and assume the results in it. There are two parts: the first investigates the g-function in more detail and reduces the possible region for (γ, Γ), and the second establishes the existence of points on the (Γ, γ) diagram with $\Gamma \neq \gamma$.

The countable state space will be denoted by I.

2. Some properties of the g-function

We recall from [2] that a Markov semigroup has property (F) if and only if there is some $t > 0$ such that $g(t) > \frac{1}{2}$. We are particularly interested in whether there are any non-(U)-semigroups with property (F). The following simple lemma will prove very useful.

LEMMA 1. *For all u and t such that $0 < u < t$ and for all i and $k \in I$, there exist states j and l such that $p_{j,i}(u) \geq p_{k,i}(t)$ and $p_{l,i}(u) \leq p_{k,i}(t)$.*

We have $p_{k,i}(t) = \sum_{h \in I} p_{k,h}(t-u)p_{h,i}(u)$. This is a convex combination of the $p_{h,i}(u)$ $(h \in I)$, and so there exist states j and l as required.

DEFINITION. *Let $S_i(t) = \sup_j p_{j,i}(t)$ and $s_i(t) = \inf_j p_{j,i}(t)$.*

COROLLARY. *For each i, $S_i(t)$ is a nonincreasing function of t, and $s_i(t)$ is nondecreasing.*

THEOREM 1. (a) *On the set $\{t: g(t) \geq \frac{1}{2}\}$, $g(t)$ is a nonincreasing function.*

(b) *A direct sum, $P(t)$, of semigroups, $P_r(t)$, each having the property that its g-function, $g_r(t)$, is continuous on the left, has $g(t) \leq \frac{1}{2}$ for all $t > 0$ unless it enjoys property (U).*

(c) *If $g(t) = m > \frac{1}{2}$, then, for each $u < t$, either $g(u) \geq m$ or $g(u) \leq 1 - m$.*

When $g(t) \geq \frac{1}{2}$, $p_{i,i}(t) \geq \frac{1}{2}$ for all i. Since the row sums of $P(t)$ are 1, each off-diagonal element is less than or equal to $\frac{1}{2}$, and so, for such values of t, $S_i(t) = p_{i,i}(t)$ for every i. Statement (a) now follows from the above corollary.

To prove (b), first note that $g(t) = \inf_r g_r(t)$. We assume that for some $t_0 > 0$, $g(t_0) = m > \frac{1}{2}$. Then $g_r(t_0) \geq m$ for all r. The proof rests on the fact that for each r, $g_r(t)$ is monotone nonincreasing on $[0, t_0]$. By (a), it will be sufficient to show that for $t \in (0, t_0)$, $g_r(t)$ is never less than $\frac{1}{2}$. Assume on the contrary that $g_r(T) < \frac{1}{2}$ for some $T \in (0, t_0)$. The semicontinuity of $g_r(t)$ implies that the set