A SURVEY ON THE MARKOV PROCESS ON THE BOUNDARY OF MULTI-DIMENSIONAL DIFFUSION

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1. Introduction

It has been observed that the Markov process on the boundary of diffusion is related to the solution of a diffusion equation in a domain D, $(\partial/\partial t)u = Au$, with Wentzell's boundary condition, $Lu = 0$. (Precise definitions of A and L are given in section 2, (2.1) and (2.3).) In fact, to obtain the solution, it is sufficient to solve $(\alpha - A)u(x) = 0$, $x \in D$ and $(\lambda - L)u(x) = \varphi(x)$, $x \in \partial D$ for sufficiently many φ on the boundary ∂D, where $\alpha \geq 0$ and $\lambda \geq 0$ are fixed. This provides a class of Markov processes on ∂D ([13], [14]).

A kind of duality between the way of obtaining the diffusion on $\bar{D}$ and the way of obtaining a process of this class naturally leads to a conjecture that the Markov process on the boundary is the trace on the boundary of the trajectory of the diffusion. Moreover, a simple example suggests that this trace is described by a time scale called the local time on the boundary $\mathfrak{t}(t, w)$ in such a way that

$$\mathfrak{x}(t, w) = x(\mathfrak{t}^{-1}(t, w), w), \tag{1.1}$$

where $x(t)$ denotes the path function of the diffusion, $\mathfrak{x}(t)$ denotes the Markov process on ∂D, and $\mathfrak{t}^{-1}(t, w)$ is the right continuous inverse of $\mathfrak{t}$ [14]. In fact, K. Sato proved that this is true in the case of reflecting diffusion with sufficient regularities ([10], [11], [13]). Such a process on the boundary had not been explicitly discussed because the boundaries of one-dimensional diffusion are too simple.

However, the concepts of Markov process and local time on the boundary of a diffusion process can also be considered apart from the setup based on elliptic operator A and boundary condition L. In fact, a well-known correspondence between excessive functions and additive functionals insures the existence of a class of additive functionals which increase when and only when $x(t, w)$ is on the boundary, and we obtain a Markov process $\mathfrak{x}(t, w)$ on ∂D by making use of such a functional as before [8], [12], [13].

From this point of view, a part of the problem Feller solved in one dimension can be formulated in the following way. Given a diffusion process M on a domain $\bar{D}$, determine the class of all diffusions whose path functions coincide with those of M before they arrive at ∂D, where jumps from the boundary are permitted. In other words, let $\mathsf{M}^{\min}$ be a diffusion whose path functions vanish