GENERAL LATERAL CONDITIONS FOR SOME DIFFUSION PROCESSES

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1. Formulation of the problem and fundamental results

1.1. Let E be a plane domain bounded by a smooth contour L, and let $v(z)$ be a smoothly varying vector field on L. Let the point $\gamma \in L$ be called exclusive if the projection of the vector $v(z)$ on the inner normal to L changes sign at the point γ. Let us say that the function $u(z)$ satisfies the boundary condition $\mathfrak{A}$ if, at each nonexclusive point z of the contour L, the derivative of u in the direction $v(z)$ is zero. We are interested in solutions of the heat conduction equation $(\partial u_t(z)/\partial t) = \Delta u_t(z)$ in the domain E, which satisfy the initial condition $u_0(z) = f(z)$ and the boundary condition $\mathfrak{A}$. More accurately, our problem is to describe the general form of the lateral conditions at exclusive points, which will, together with the initial and boundary conditions, define a unique solution $u_t(z)$ of the heat conduction equation, wherein: (a) $u_t(z) \geq 0$ if $f(z) \geq 0$; (b) $\|u_t\| \leq \|f\|$ (we understand $\|f\|$ to be sup $|f(z)|$ in the union E^* of the domain E and the set of all nonexclusive points of the contour L). (An analogous problem for the system of differential equations of Kolmogorov which describes Markov processes with countable phase space was studied by W. Feller [4]. However, Feller considered only a special class of supplementary conditions corresponding to "continuous exit" from the boundary. The supplementary conditions we found cover the most general case.)

In terms of probability theory the problem may be stated as follows. The heat conduction equation, together with the boundary condition $\mathfrak{A}$, prescribes a Brownian motion process in the domain E with reflection from the boundary in the domain E. The behavior of the trajectories after hitting an exclusive point of the boundary is not determined here. The problem is to describe all possible kinds of such behavior.

It is more convenient to pose and solve the problem in the terminology of semigroups of linear operators. Let $\mathcal{E}$ be some set and $\mathcal{B}$ some σ-algebra of subsets of $\mathcal{E}$. Let $B = B(\mathcal{E})$ the space of all bounded $\mathcal{B}$-measurable functions on $\mathcal{E}$ with the norm $\|f\| = \sup |f(z)|$. The family of linear operators T_t, $(t > 0)$, operating in the space B and satisfying the following conditions:

(1.1.A) $T_tf \geq 0$, if $f \geq 0$,

(1.1.B) $\|T_tf\| \leq \|f\|$,

(1.1.C) $T_sT_t = T_{s+t}$ for any s, $t > 0$,

is called a *Markov semigroup* in the space $\mathcal{E}$.