FIRST EXIT TIMES FROM A SQUARE ROOT BOUNDARY

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1. Introduction

The thing that motivated the present paper was a curious observation by Blackwell and Freedman [1]. Let $X_1, X_2, \cdots$, be independent ± 1 with probability $\frac{1}{2}$, $S_n = X_1 + \cdots + X_n$, and $T_c = \min \{n; S_n > c\sqrt{n}, c\sqrt{n} \geq 1\}$. Then for all $c < 1$, $ET_c < \infty$, but $c \geq 1$ implies $ET_c = \infty$. In order to understand this better, I wanted to calculate the asymptotic form of $P(T_c > n)$ for large n. It was reasonable to conjecture that $P(T_c > n) \sim \alpha n^{-\beta}$, not only for coin-tossing random variables but for a large class. The first step in the proof of this was to verify the result for Brownian motion. This is done in the second paragraph and follows easily from known results.

To go anywhere from there, one would like to invoke an invariance principle. But the difficulty is clear—for general identically distributed, independent r.v. $X_1, X_2, \cdots$ with $EX_1 = 0$, $EX_1^2 = 1$, the most one could hope for is that $P(T_c > n) \sim \alpha n^{-\beta}$ where β is the same for all distributions, but α depends intimately on the structure of the process. Hence, this is not a situation in which the usual invariance principle is applicable. But the result does hold for all distributions such that $E|X_1|^3 < \infty$, and it is proved by using results of Prohorov [2] which give estimates of the rate of convergence of the relevant invariance theorem. This proof is carried out in the third paragraph.

A dividend of the preceding proof is collected. The conclusion is that

$$P(S_n < \xi\sqrt{n} \,|\, |S_k| < c\sqrt{k}, k = 1, \cdots, n) \to G(\xi) \tag{1.1}$$

where $G(\xi)$ is the corresponding distribution for Brownian motion.

The related works that I know of are the interesting results given by Strassen at this symposium [3], and by Darling and Erdös [4]. There is also a very recent article by Chow, Robbins, and Teicher [5] which generalizes the result of Blackwell and Freedman.

2. First exit distribution for Brownian motion

Take $\xi(t)$ to be Brownian motion with $E\xi(t) = 0$, $E\xi^2(t) = t$. Define $T_c^* = \inf \{t; \xi(t) \geq c\sqrt{t}, t \geq 1\}$, that is, T_c^* is the first exit time past $t = 1$.

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