ON COMBINATORIAL METHODS IN THE THEORY OF STOCHASTIC PROCESSES

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1. Introduction

The main object of this paper is to prove a simple theorem of combinatorial nature and to show its usefulness in the theory of stochastic processes. The theorem mentioned is as follows.

THEOREM 1. *Let* $\varphi(u)$, $0 \le u < \infty$, *be a nondecreasing step function satisfying the conditions* $\varphi(0) = 0$ *and* $\varphi(t + u) = \varphi(t) + \varphi(u)$ *for* $u \ge 0$ *where* t *is a finite positive number. Define*

$$\delta(u) = \begin{cases} 1 & \text{if} \quad v - \varphi(v) \ge u - \varphi(u) \quad \text{for} \quad v \ge u, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then

$$\int_0^t \delta(u)\, du = \begin{cases} t - \varphi(t) & \text{if} \quad 0 \le \varphi(t) \le t, \\ 0 & \text{if} \quad \varphi(t) \ge t. \end{cases} \tag{2}$$

PROOF. If $\varphi(t) > t$, then $\delta(u) = 0$ for every u, and thus the theorem is obviously true.

Now consider the case $0 \le \varphi(t) \le t$. For $u \ge 0$ define $\psi(u) = \inf \{v - \varphi(v)$ for $v \ge u\}$. We have $\psi(u) \le u - \varphi(u)$, and $\psi(u) = u - \varphi(u)$ if and only if $\delta(u) = 1$ (compare figures 1, 2, 3).

It is clear that $\psi(u + t) = \psi(u) + t - \varphi(t)$ for $u \ge 0$ and that $0 \le \psi(v) - \psi(u) \le v - u$ for $0 \le u \le v$. Thus $\psi'(u)$ exists for almost all u, $0 \le \psi'(u) \le 1$, and

$$\int_0^t \psi'(u)\, du = \psi(t) - \psi(0) = t - \varphi(t) \tag{3}$$

because $\psi(u)$ is a monotone and absolutely continuous function of u. We also note that $\varphi(u + 0) = \varphi(u)$ and $\varphi'(u) = 0$ for almost all u.

First we prove that

$$\psi'(u) \le \delta(u) \qquad \text{for almost all } u. \tag{4}$$

If $\psi'(u)$ exists and if $\psi'(u) = 0$, then (4) evidently holds. Now we shall prove

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