SUPPORTS OF CONVOLUTIONS OF IDENTICAL DISTRIBUTIONS

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1. Introduction

It has long been known that the convolution of two singular univariate distributions can be absolutely continuous, and that this can hold even if the distributions are concentrated on sets of Hausdorff dimension zero. However, the problem is more complicated if the convolvants are required to be identical.

What is done in this paper is to give simple examples of convolutions of singular distributions, even concentrated on sets of Hausdorff dimension zero, which are absolutely continuous. Furthermore, if we define the "dimension" of a distribution to be the smallest Hausdorff dimension of a set with probability one and the distribution is singular, and ∞ if the distribution is absolutely continuous, then if monotonically nondecreasing functions f, g, h are given from $(0, \infty)$ to $[0, 1] \cup \{\infty\}$, such that for each t, $f(t) \leq h(t)$ and $g(t) \leq h(t)$, there exist infinitely divisible distributions F and G such that the dimension of F^t is $f(t)$, the dimension of G^t is $g(t)$, and the dimension of $F^t * G^t$ is $h(t)$. Furthermore, the measure in the Lévy-Khintchine representation of F is purely discrete, and that of G is purely singular. This generalizes results of Tucker [4] and Rubin [3]. We can even insist that the distributions F^t, G^t, $F^t * G^t$ are pure, that is, there is no nonzero component of smaller dimension.

2. An example

For our first example, let

$$X_i = \sum_{j=1}^{\infty} 2^{-j} A_{ij}, \qquad i = 1, 2, \tag{1}$$

where the A_{ij} are independent zero or one random variables and $P(A_{ij} = 1) = p_j$. Then it is known that the distribution of X_i is absolutely continuous if and only if $\sum (p_j - \frac{1}{2})^2 < \infty$. We now show that if $\sum (p_j - \frac{1}{2})^4 < \infty$, the distribution F of $X_1 + X_2$ is absolutely continuous.

We first observe that, since $X_1 + X_2$ is a sum of independent discrete random variables, its distribution is pure [5]. Hence, for F to be absolutely continuous,

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