ON REGULAR VARIATION AND LOCAL LIMIT THEOREMS

WILLIAM FELLER
PRINCETON UNIVERSITY

1. Introduction

Recent work on limit theorems in probability is marked by two tendencies. The old limit theorems are being supplemented and sharpened by a variety of so-called local limit theorems (which sometimes take the form of asymptotic expansions). Even more striking is the increasing role played by functions of regular variation. They made their debut in W. Doblin's pioneer work of 1940 where he gave a complete description of the domains of attraction of the nonnormal stable distributions. A long series of investigations started by E. B. Dynkin and continued by J. Lamperti, S. Port, and others have shown that essential limit theorems connected with renewal theory depend on regular variation. The same is true of the asymptotic behavior of the maximal term of a sequence of independent random variables and of D. A. Darling's theorems concerning the ratio of this term to the corresponding partial sum.

It seems that each of these problems still stands under the influence of its own history and that, therefore, a great variety of methods is used. Actually a considerable unification and simplification of the whole theory could be achieved by a systematic exploitation of two powerful tools: J. Karamata's beautiful theory of regular variation and the method of estimation introduced by A. C. Berry in his well-known investigation of the error term in the central limit theorem.

[It seems that proofs of Karamata's theorems can be found only in his paper of 1930 in the Rumanian journal *Mathematica* (Vol. 4), which is not easily accessible. For purposes of probability theory, one requires a generalization from Lebesgue to Stieltjes integrals. A streamlined version is contained in the forthcoming second volume of my *Introduction to Probability Theory*, but this book does not contain the inequalities derived in the sequel.]

Berry's method is of wide applicability and not limited to the normal distribution. It leads to an estimate for the discrepancy between distributions in terms of the discrepancy between the corresponding characteristic functions. In the case of the normal distribution, the latter discrepancy can be estimated in terms of the moments, and the theory of regular variation leads readily to similar

Research supported by Army Research Office (Durham) Project in Probability at Princeton University.