ALMOST SURE BEHAVIOR OF SUMS OF INDEPENDENT RANDOM VARIABLES AND MARTINGALES

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1. Introduction and summary

Toss a fair coin independently n times and let S_n be the number of heads minus the number of tails. Khintchine [26] proved the law of the iterated logarithm

$$\limsup_n (2n \log\log n)^{-1/2} S_n = 1, \qquad \text{a.s.} \tag{1}$$

Thus if the supremum of an empty set of natural numbers is understood to be 0, then

$$\hat{T} := \sup \{n: (2n \log\log n)^{-1/2} S_n \geq c\} \tag{2}$$

is a random variable for $c > 1$ representing the speed of the upper half of the law of the iterated logarithm.

In the first part of this paper we will be mainly concerned with the distribution of $\hat{T}$ and of similar random variables. Our research has been initiated by a remark of Professor Alfred Rényi to the effect that $E\hat{T} = \infty$. Here is a simple proof:

$$\begin{aligned} E\hat{T} &\geq E\,(\text{number of times } n \text{ such that } (2n \log\log n)^{-1/2} S_n \geq c) \\ &= \sum_n \Pr\{(2n \log\log n)^{-1/2} S_n \geq c\} = \infty \end{aligned} \tag{3}$$

by normal approximation.

Unfortunately, this argument seems to have a very limited scope. Replace, for instance, $(2n \log\log n)^{-(1/2)}$ in the definition of $\hat{T}$ by $(2n(\log n + \log\log n))^{-(1/2)}$ and put $c = 1$. Then $E\hat{T} = \infty$ (as will follow from corollary 4.7 of this paper), but

$$\sum_n \Pr\{(2n(\log n + \log\log n))^{-1/2} S_n \geq 1\} < \infty. \tag{4}$$

Moreover, the argument (3) fails if applied to moments other than the mean.

In order to state our first result in a more suggestive way, let us introduce a

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