F-PROCESSES

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1. Introduction

In [1] Doob associated with any sequence $\{X_n\}$ of random variables two new sequences $\{M_n\}$ and $\{W_n\}$ in such a way that $X_n = M_n + W_n$, $\{M_n\}$ is a martingale, and finally $\{W_n\}$ is a.s. monotone noncreasing in n if and only if $\{X_n\}$ is a supermartingale. Doob noted that an analogous decomposition in the continuous parameter case did not seem easy to obtain.

Recently, two interesting works have dealt with the continuous parameter problem. Consider the case where the parameter varies over a compact interval $[0, a]$. First P. A. Meyer [6], [7] showed that a right-continuous supermartingale $\{X_t\}$ will satisfy $X_t = M_t - A_t$ for some martingale $\{M_t\}$ and some process $\{A_t\}$ which has almost all sample functions right-continuous, monotone increasing, vanishing at $t = 0$ and which satisfies $E|A_a| < \infty$ if and only if $\{X_t\}$ satisfies a certain mild integrability condition. A process satisfying said integrability condition Meyer refers to as belonging to *class* D.

Then D. L. Fisk, following up ideas introduced by Herman Rubin in an invited address at the I.M.S. meetings at the University of Oregon in 1956, considered a class of processes, called F-processes below, and showed in [3] that an F-process with continuous sample paths could be decomposed as $X_t = M_t + W_t$, where $\{M_t\}$ is a martingale and $\{W_t\}$ has almost every sample function of bounded variation, the total variation even having a finite expectation, if and only if $\{X_t\}$ belongs to class D. Fisk's methods depend on the assumption of continuity; on the other hand, Meyer's methods depend on dealing with supermartingales, rather than just F-processes.

The present paper grew out of a desire to prove a decomposition theorem for F-processes without assuming continuity. Our main result is that for right-continuous F-processes the desired decomposition exists if and only if the process belongs to class D. The assumption of right-continuity is fairly harmless, as will be seen below. Our proof makes heavy use of ideas of both Meyer and Fisk.

We will say $X = \{X_t, \mathfrak{F}_t, \Gamma\}$ is a *stochastic process* if $\Gamma \subseteq [0, \infty)$, for each $t \in \Gamma$, X_t is a random variable, $\mathfrak{F}_t$ is a Borel field of events, $s < t$ and $s \in \Gamma$, $t \in \Gamma$ imply $\mathfrak{F}_s \subseteq \mathfrak{F}_t$, and finally X_t is measurable with respect to $\mathfrak{F}_t$.

Let $X = \{X_t, \mathfrak{F}_t, [0, a]\}$ be a stochastic process, and let a be a positive number. Then X is an *F-process* if $E|X_t| < \infty$ for all t, and there exists a number K such that for every partition $t_0 < t_1 < \cdots < t_n$ of $[0, a]$,

$$(1) \qquad E\left\{\sum_{j=0}^{n-1} |E\{(X_{t_{j+1}} - X_{t_j})|\mathfrak{F}_{t_j}\}|\right\} \leq K.$$